Manifolds and Riemannian Geometry

Steinmetz Symposium

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What is Differential Geometry?

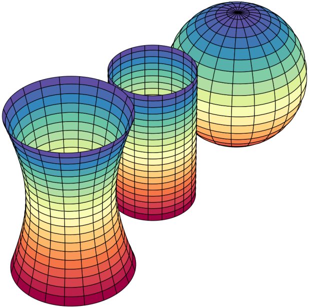

Figure: Surfaces of different curvature.

Differential geometry studies surfaces and how to distinguish between them— and other such objects unique to each type of surface.

Defining a Manifold

We begin by defining a manifold.

Definition

A topological space M is *locally Euclidean of dimension n* if every point $p \in M$ has a neighborhood U such that there is a homeomorphism ϕ from U to an open subset of $\mathbb{R}^n$. We call the pair $(U, \phi : U \to \mathbb{R}^n)$ a *chart*.

Definition

A topological space M is called a *topological manifold* if the space is Hausdorff, second countable, and locally Euclidean.

The Riemann Metric

Geometry deals with lengths, angles, and areas, and "measurements" of some kind, and we will encapsulate all of these into one object.

Definition

The *tangent space* $T_p(M)$ is the vector space of every tangent vector to a point $p \in M$.

Definition (Important!)

A *Riemann metric* on a manifold to each point p in M of an inner product $\langle\ ,\ \rangle_p$ on the tangent space T_pM; moreover, the assignment $p \mapsto \langle\ ,\ \rangle$ is required to be C^∞ in the following sense: if X and Y are C^∞ vector fields on M, then $p \mapsto \langle X_p, Y_p\rangle_p$ is a C^∞ function on M. A *Riemann manifold* is a pair $(M, \langle\ ,\ \rangle)$ consisting of the manifold M together with the Riemann metric $\langle\ ,\ \rangle$ on M.

Existence of a Riemann Metric

Theorem

On every manifold M there is a Riemann metric.

Definition

Let $\{U_\alpha\}_{\alpha \in A}$ be an open cover of a manifold M. We define a *partition of unity subordinate to* $\{U_\alpha\}$ as the collection of functions

$$\rho_\alpha : M \to \mathbb{R}, \quad \alpha \in A$$

satisfying

1. $\mathrm{supp}\rho_\alpha \subset U_\alpha$ for all α,
2. the collection of supports, $\{\mathrm{supp}\rho_\alpha\}_{\alpha \in A}$, is locally finite,
3. $\sum_{\alpha \in A} \rho_\alpha = 1$.

Existence of a Riemann Metric

Proof

We will construct the Riemann metric using a partition of unity subordinate to some coordinate chart. We will choose $\{U_\alpha, \phi_\alpha\}$ with its associated local coordinates $x^1, ..., x^n$. For a vector field $X = \sum a^i \partial_i$ and $Y = \sum b^j \partial_j$, we define the Riemann metric on U_α as

$$\langle X, Y \rangle_\alpha = \sum a^i b^j \partial_i \partial_j = \sum a^i b^j \delta_{ij} = \sum a^i b^i.$$

Let $\{\rho_\alpha\}$ be a partition of unity subordinate to $\{U_\alpha\}$. By the local finiteness property of the partition of unity, for every point $p \in M$ there exists a neighborhood U_p at p for which the collection $\{\operatorname{supp} \rho_\alpha\}$ has finitiely many nontrivial ρ_α. Thus, $\sum_{\alpha \in A} \rho_\alpha \langle , \rangle_\alpha$ is a finite sum on U_p. We then want to show that this sum is an inner product on $T_p M$. It is easy to show that it is symmetric and bilinear, coming down to the symmetry and bilinearity of the individual inner products in the sum.

Existence of a Riemann Metric

Proof

It remains to show positive-definiteness. Let $X \in T_pM$ be a nonzero tangent vector, and define A_p as:

$$A_p = \{\alpha \in A \,|\, \text{supp}\, \rho_\alpha \text{ is nontrivial in } U_p\}.$$

Since for all $\alpha \in A$ each inner product $\langle X, X \rangle_\alpha > 0$ from the positive-definiteness of the inner product. We define the minimum

$$m = \min_{\alpha \in A_p} \{\langle X, X \rangle_\alpha\} > 0.$$

Then at p, we have

$$\sum_{\alpha \in A} \rho_\alpha \langle X, X \rangle_\alpha = \sum_{\alpha \in A_p} \rho_\alpha \langle X, X \rangle_\alpha \geq m \sum_{\alpha \in A} \rho_\alpha = m > 0,$$

as desired. $\square$

Taking Derivatives on Manifolds

Definition

Now, we will start building the machinery necessary to relate the metric to curvature.
The *directional derivative* at a point $p \in \mathbb{R}^n$ of a smooth vector field Y, defined by $Y = \sum b^i \partial_i$ on $\mathbb{R}^n$ in a direction X_p is defined to be

$$D_{X_p} Y = \sum (X_p b^i) \frac{\partial}{\partial x^i}\bigg|_p$$

Properties of the Directional Derivative

For $X, Y \in \mathfrak{X}(\mathbb{R}^n)$, the directional derivative $D_X Y$ satisfies the following properties

1. $D_X Y$ is $\mathcal{F}$-linear in X and $\mathbb{R}$-linear in Y;
2. if f is a smooth function on $\mathbb{R}^n$, then

$$D_X(fY) = (Xf)Y + fD_X Y$$

Taking Derivatives on Manifolds

Definition

An *Affine Connection* on a manifold M is an $\mathbb{R}$-bilinear map

$$\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$$

written $\nabla_X Y$ for $\nabla(X, Y)$ satisfying the two properties below: if $\mathcal{F}$ is the ring of smooth functions on M, then for all $X, Y \in \mathfrak{X}(M)$,

1. $\nabla_X Y$ is $\mathcal{F}$-linear in X,
2. $\nabla_X Y$ satisfies the Leibniz rule in Y: for $f \in \mathcal{F}$,

$$\nabla_X(fY) = (Xf)Y + f\nabla_X Y.$$

The Riemann Curvature Tensor

Definition

We can define the *Riemann curvature tensor* associated with an affine connection ∇ on M as

$$R(X, Y) = [\nabla_X, \nabla_Y] - \nabla_{X,Y]} = \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]}.$$

Amazingly, the Riemann curvature tensor is $\mathcal{F}-$linear in all of its arguments.

Theorem (HUGE!)

On a Riemannian manifold there is a unique Riemannian connection, defined by

$$\begin{aligned} 2\langle \nabla_X Y, Z \rangle = & X\langle Y, Z\rangle + Y\langle Z, X\rangle - Z\langle X, Y\rangle \\ & - \langle X, [Y, Z]\rangle + \langle Y, [Z, X]\rangle + \langle Z, [X, Y]\rangle . \end{aligned}$$

Curvature in 3 Dimensions

We define the curvature function for a normal section of a surface $\gamma(s)$ at a point p with respect to a normal vector N_p as

$$\kappa(X_p) = \langle \gamma''(0), N_p \rangle \,.$$

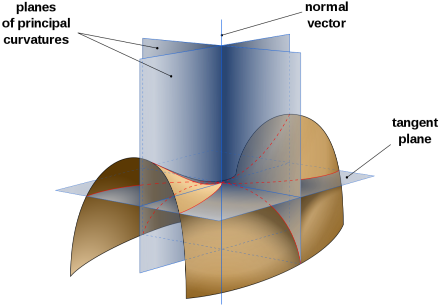

Figure: Normal sections of a Surface in 3 dimensions.

Curvature in 3 Dimensions

We define the minimum and maximum of the normal curvature, κ_1 and κ_2, to be the *principle curvatures* of a surface M at p. The *Gauss curvature* of M at p is the product of the two principle curvatures.

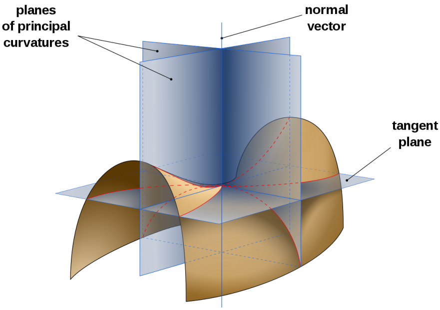

Figure: Normal sections of a Surface in 3 dimensions.

Curvature in 3 Dimensions

We can relate a special type of affine connection to the Gauss curvature of a surface.

Definition

Let p be a point on a surface M in $\mathbb{R}^3$ and let N be a smooth, unit normal vector field on M. For any tangent vector $X_p \in T_pM$, we define

$$L_p(X_p) = -D_{X_p}N.$$

This is the *shape operator* to a surface M at a point p, and is a map from the tangent space of a surface to itself, and has the principle curvatures of p as its eigenvalues. When represented as a matrix, the determinant of the shape operator is the Gauss curvature at p.

Curvature in 3 Dimensions

Finally, we will relate the Riemann curvature tensor to the shape operator and then to the Gauss curvature.

Theorem

Let M be an oriented surface in $\mathbb{R}^3$, with Riemann connection ∇. Let R be the associated curvature tensor of the Riemann connection and L the shape operator on M. Then for $X, Y, Z \in \mathfrak{X}(M)$, we have

$$R(X, Y)Z = \langle L(Y), Z \rangle L(X) - \langle L(X), Z \rangle L(Y)$$

Gauss's Theorema Egregium

Let M be a surface in $\mathbb{R}^3$ and p a point in M. If e_1, e_2 is an orthonormal basis for the tangent plane T_pM, then the Gaussian curvature K_p of M at p is

$$K_p = \langle R_p(e_1, e_2)e_2, e_1 \rangle .$$

Proof of the Theorema Egregium

Proof

We can express the Gauss curvature of a surface, K_p, in terms of the shape operator:

$$K_p = \langle L(e_1), e_1 \rangle \langle L(e_2), e_2 \rangle - \langle L(e_1), e_2 \rangle \langle L(e_2), e_1 \rangle .$$

Via the Gauss curvature equation, we have

$$R_p(e_1, e_2)e_2 = \langle L(e_2), e_2 \rangle L(e_1) - \langle L(e_1), e_2 \rangle L(e_2).$$

We then take the inner product on the above, which gives

$$\begin{aligned} \langle R_p(e_1, e_2)e_2, e_1 \rangle &= \langle L(e_1), e_1 \rangle \langle L(e_2), e_2 \rangle - \langle L(e_1), e_2 \rangle \langle L(e_2), e_1 \rangle \\ &= K_p. \end{aligned}$$

References

[1] **L.W. Tu**. An Introduction to Manifolds. Universitext. Springer New York, 2010.

[2] **L.W. Tu**. Differential Geometry: Connections, Curvature, and Characteristic Classes. Graduate Texts in Mathematics. Springer International Publishing, 2017.

Image 1 accessed from Wikipedia "Gauss Curvature"

Image 2 accessed from Wikipedia "Normal Plane (Geometry)".