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Comparing Newtonian Mechanics, Special Relativity and General Relativity for Motion Near a Mass

Aqeel Ahmed
Union College - Schenectady, NY

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Abstract

With both analytical and computational methods, we modeled an object's orbit around a massive object using Newtonian orbits, using special relativistic orbits, and finally using general relativistic orbits. We know that when an object orbits around a massive object, there are certain velocities and radii where the object will no longer orbit and fall into the massive object, in a correct orbit model we should expect to see this feature. This distinctive feature is not captured by our Newtonian model. With special relativity, the feature appears, however, special relativity contains disparities that do not correctly model the effect of gravity. With general relativity, the feature is seen and the disparities from special relativity are no longer apparent. The general relativistic model correctly matches our predictions of the orbit around a massive object, the object either falls into, is bounded by, or is unbounded by the massive object, the model also contains a space where the physics starts to break down, inside the event horizon.

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Chapter 1

Introduction

Isaac Newton built upon Johannes Kepler's laws of planetary motion to formulate the universal law of gravitation in the late 17th century. This law states that all objects of matter are attracted to each other by an invisible force known as gravity. The force is proportional to the product of their masses and inversely proportional to the square of the distance between them. Newton's law mathematically and accurately described the force of gravity that exists between any two objects. This served as the cornerstone for understanding gravity, and it is still being used today, even for navigating spacecraft [1].

However, there were ways that his law fell short. The earliest observation that challenged Newton's law was the precession of Mercury's orbit. It was observed that the perihelion of Mercury's orbit advanced slightly over time. This discrepancy in Mercury's orbit could not be accounted for according to the universal law of gravitation. Newton's law starts to break down when considering extreme speeds and extreme gravity, such a scenario exists when speeds are comparable to the speed of light and where objects are extremely massive. Other phenomena that are not accorded for in Newton's law are the effect of gravitational lensing and the existence of gravitational waves. Apart from these errors, for nearly 220 years, Newton's universal law of gravitation stood as the fundamental pillar of our understanding of gravity [1] [2].

In 1905, Albert Einstein presented his theory of special relativity which revolutionized our understanding of space and time. It introduced two key principles: the principle of relativity and the invariance of the speed of light. The theory of relativity states that the laws of physics are the same for all observers in constant motion relative to each other. The constancy of the speed of light says that the speed of light in a vacuum is constant and independent of the motion of the observer. This challenged the classical notion of absolute time and introduced the concept of time dilation, which states that time passes more slowly for objects in motion relative to an observer at rest. Special relativity also offers a relativistic relationship between mass, momentum, and energy. Although special relativity gave us a better understanding of the speed of light and space-time, we were left with more questions about the intricate workings of gravity. In 1915, Einstein presented an answer to those questions [3].

The general theory of relativity expanded on special relativity and provided a new understanding of gravity. The equivalence principle is a foundational concept in general relativity that describes the relationship between gravitational acceleration and inertial motion. It states that in a sufficiently small region of space-time, the effects of gravity are indistinguishable from the effects of acceleration. That is to say, $F_g = F_a$, or $m\boldsymbol{g} = m\boldsymbol{a}$ which implies $\boldsymbol{g} = \boldsymbol{a}$. However, unlike Newtonian mechanics, gravity is not a force but rather a curvature of space-time caused by mass. This curvature acts as an underlying characteristic that affects all particles the same way. It is important to note that mass is a form of energy and therefore

energy can also serve as a gravitational source as mass does [4].

The ultimate goal of this paper is to analyze the specific differences that appear between the Newtonian form of gravity, the special relativistic form of gravity, and the general relativistic form of gravity. Specifically, we will be looking at how the orbit of a point-like object changes when orbiting around a fixed point-like massive object. In both Newtonian and special relativity, the fixed point-like massive object behaves as a normal object would, however, in general relativity, the object behaves as a black hole would. In chapters two, three, and four, we go through the theory behind the ideas we use in our main calculations. These calculations occur in chapters five, six, and seven, where we present the resulting orbits.

Part I

Theory

In the following chapters, we will introduce the background theory for the problem we want to look at. This will include what invariance means for certain transformations, defining tensors, looking at curved space-time, and ultimately solving Einstein's equation in order to derive the Schwartzschild metric. We split apart these theory chapters into Newtonian, special relativity, and general relativity cases. This is to connect the ideas presented in each chapter with the specific laws of physics we focus on in that chapter.

Chapter 2

Theory: Newtonian

In this chapter, we introduce the theoretical aspects necessary for understanding the physics that we will use when introducing more complicated setups. This chapter is referred to as the Newtonian chapter as it primarily focuses on concepts that can be comprehended at a fundamental level within the framework of Newtonian physics. The ideas and principles presented here are accessible and graspable from a Newtonian standpoint.

We will start by establishing some facts about coordinate systems that will be useful for understanding the framework of Newtonian physics. This is the beginning of a discussion that will be extended when we take into account special relativity, where instead of space, we start discussing space-time, and take into account general relativity, where instead of a flat background, we will start with a curved background.

The principle of least action is a principle in physics that underlies various branches of classical mechanics and field theory. It states that the path taken by a physical system between two points in space-time is the one for which a specific quantity, known as the action, is minimized. According to the principle, a physical system will follow a path that minimizes the action. This path is known as the "path of least action" or "trajectory of extremal action." The principle states that the actual path taken by a system is one for which the action, a quantity defined by the Lagrangian is minimized. We are going to use this concept to understand special and general relativity.

2.1 Changing Axis Orientation

Coordinate axes are something that we introduce into mathematical descriptions of the world. As a result, it makes sense that the laws of physics do not care about the axes we orient ourselves to. Consequently, we should be able to see that the laws of physics are unaffected by choosing different axes, and in fact, we can think of this as a necessary requirement for any valid physical law. In order to have that discussion, we want to start by establishing the ways vectors transform under coordinate rotations. To begin, we explore how the components of the vector $\boldsymbol{v}$ change when the axis is rotated by an angle θ.

We will be using a three-dimensional space with coordinates (x_1, x_2, x_3), so that a vector can be constructed from three components (v_1, v_2, v_3) as $\boldsymbol{v} = v_1\boldsymbol{e}_1 + v_2\boldsymbol{e}_2 + v_3\boldsymbol{e}_3$. We will be rotating around x_3 axis by an amount θ as shown in figure 2.1.

Looking at figure 2.1, we can see that rotating the axis causes the vector ($\boldsymbol{v} = v_1\boldsymbol{e}_1 + v_2\boldsymbol{e}_2 + v_3\boldsymbol{e}_3$) to be

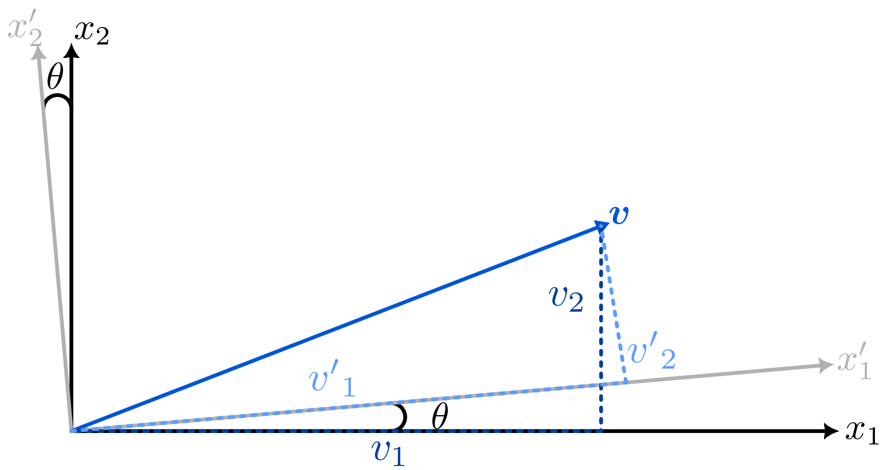


Figure 2.1: Diagram of the rotation of the axis for a vector $\boldsymbol{v}$ by θ degrees.

defined by new components ($\boldsymbol{v} = v'_1\boldsymbol{e}'_1 + v'_2\boldsymbol{e}'_2 + v'_3\boldsymbol{e}'_3$). The values of both v'_1 and v'_2 can be found as

$$v'_1 = v_2 \sin\theta + v_1 \cos\theta, \qquad v'_2 = v_2 \cos\theta - v_1 \sin\theta, \tag{2.1}$$

and for $v_3{}'$, we have that $v_3{}' = v_3$.

We can also give these transformations using matrix multiplication as

$$\begin{bmatrix} v'_1 \\ v'_2 \\ v'_3 \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}, \tag{2.2}$$

or, in index notation

$$v'_i = \sum_{j=1}^{3} R_{ij} v_j.$$

Consider the dot product $\boldsymbol{v} \cdot \boldsymbol{w}$ between two vectors v and w. This should not change when we rotate the coordinate axes, since by the "geometric definition" of the dot product, we have $\boldsymbol{v} \cdot \boldsymbol{w} = ||\boldsymbol{v}||\,||\boldsymbol{w}|| \cos\phi$, where ϕ is the angle between them. The lengths of the two vectors and the angle between them should remain unchanged regardless of the orientations of the coordinate axes. This fact can be demonstrated explicitly using the rotation equations 2.1. We can plug in these equations into $v'_1 w'_1 + v'_2 w'_2 + v'_3 w'_3$, to give us

$$v'_1 w'_1 + v'_2 w'_2 + v'_3 w'_3 = v_2 w_2 \sin^2\theta + v_1 w_1 \cos^2\theta + v_2 w_2 \cos^2\theta + v_1 w_1 \sin^2\theta + v_3 w_3,$$

with some more simplifying we can see that

$$v_1 w_1 + v_2 w_2 + v_3 w_3 = v'_1 w'_1 + v'_2 w'_2 + v'_3 w'_3. \tag{2.3}$$

From equations 2.1, it also becomes apparent that the preservation of vector magnitudes can be seen as a special case of this principle, where the magnitude of a vector is given by $||\boldsymbol{v}|| = \sqrt{\boldsymbol{v} \cdot \boldsymbol{v}}$. In the case that both vectors are the same we get the following, where we have the magnitude of the vectors squared,

$$v_1{}^2 + v_2{}^2 + v_3{}^2 = v'_1{}^2 + v'_2{}^2 + v'_3{}^2, \tag{2.4}$$

which can be seen from equation 2.3 [5].

2.1.1 Tensors

While certain quantities in nature are represented as vectors (e.g., velocity, force), others are scalar quantities (e.g., speed, energy, mass, time). Scalars remain unaffected by rotations ($c' = c$). However, there are more complex objects that undergo sophisticated transformations. For instance, the moment of inertia is a tensor with two indices that transforms according to the equation

$$I'_{ij} = \sum_{k=1}^{3}\sum_{l=1}^{3} R_{ik}R_{jl}I_{kl}.$$

We also observe that

$$I'_{ij} = \sum_{i=1}^{3}\sum_{j=1}^{3} I_{ij}w_i v_j,$$

remains unchanged, or is invariant. Which comes from a generalization of the idea of a dot product. These special types of objects are called “tensors”. In general, tensors can be combined through multiplication and “index summation” to create objects that do not undergo transformations [4].

2.2 The Principle of Least Action

The action principle is a fundamental principle in physics that provides a powerful framework for understanding the behavior of physical systems. It states that the path taken by a system between two points in space is the one that minimizes a quantity called the action. The principle of least action does not replace Newton’s laws of motion, instead, it provides an alternative, equivalent formulation of classical mechanics. Both approaches describe the behavior of physical systems but from different perspectives. Newton’s laws of motion, specifically his second law ($\boldsymbol{F} = m\boldsymbol{a}$), describes the relationship between the forces acting on a body, its mass, and the resulting acceleration. Newton’s laws are based on the concept of force and they provide a direct way to calculate the motion of objects under the influence of external forces. On the other hand, the principle of least action takes a more abstract and holistic viewpoint. It considers the entire path of a system between two points in space-time, rather than focusing on individual forces at each moment. In Newtonian physics, the action of a system is defined as the integral of a Lagrangian function between two points in time taken over time [6].

2.2.1 Derivation of the Euler-Lagrange Equation

With this principle, we can derive the Euler-Lagrange equation which must be satisfied in order to extremize the action. In this context, it will essentially replace Newton’s second law. We assume that we have a single particle traveling in one dimension, with motion described by some time-dependent function $x(t)$, where we fix the initial and final location of the particle. For our situation, we will derive the Euler-Lagrange equation by extremizing the following functional:

$$S[x(t)] = \int_0^T \mathcal{L}(x(t), \dot{x}(t), t)\ dt, \tag{2.5}$$

where $\mathcal{L}$ represents the Lagrangian, a function of position, velocity, and time. If we know a particle satisfies $x(0) = x_0,\ x(T) = x_T$, where these values are known, the path it will take $(x(t))$ will "extremize" the action S.

If we had a function $f(x)$ we wanted to extremize, we would set $f'(\tilde{x}) = 0$, where $\tilde{x}$ is the location of the extremum. Or equivalently,

$$f(x) = f(\tilde{x}) + f'(\tilde{x})\underbrace{(x - \tilde{x})}_{\varepsilon} + \frac{1}{2}f''(\tilde{x})\underbrace{(x - \tilde{x})^2}_{\varepsilon^2}. \tag{2.6}$$

Here, we Taylor expanded near $\tilde{x}$, and this requires the linear term to vanish. The purpose of the preceding explanation is to highlight an important connection: the process of setting $f'(x_0) = 0$ to determine the location of an extremum is equivalent to identifying x_0 as a minimum when the "linear term" disappears.

In figure 2.2, $\tilde{x}(t)$ represents the path that we want. The object $x(t)$ represents something *close* to the path we want. The term $\varepsilon(t)$ represents a small variation that when added to $\tilde{x}(t)$ will give us $x(t)$. We are only

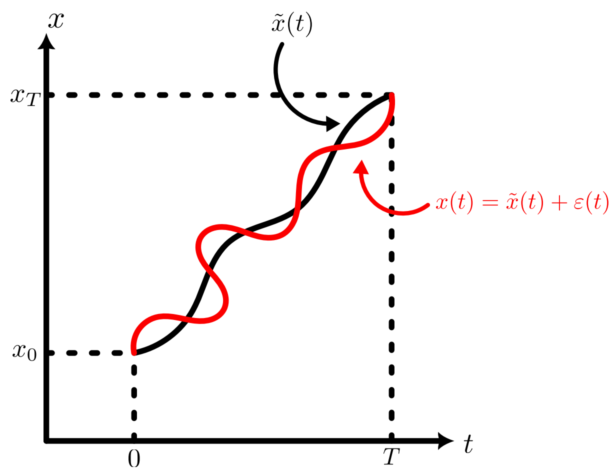


Figure 2.2: Diagram a path $\tilde{x}(t)$ with $x(t)$ representing a path is varied from $\tilde{x}(t)$ slightly.

going to consider perturbations that share the known end-points of the function, so that $x(0) = \tilde{x}(0) = x_0$, and therefore $\varepsilon(0) = 0$. Similarly for the other endpoint, we have $x(T) = \tilde{x}(T) = x_T$, which implies $\varepsilon(T) = 0$. Now we expand

$$S[\tilde{x}(t) + \varepsilon(t)] = \int_0^T \mathcal{L}\left(\tilde{x}(t) + \varepsilon(t), \dot{\tilde{x}}(t) + \dot{\varepsilon}(t), t\right) dt, \tag{2.7}$$

so that it becomes

$$S[\tilde{x}(t) + \varepsilon(t)] = S[\tilde{x}(t)] + \int_0^T \left(\frac{\partial \mathcal{L}}{\partial x}\varepsilon(t) + \frac{\partial \mathcal{L}}{\partial \dot{x}}\dot{\varepsilon}(t)\right) dt.$$

We want to make the second term of this equation vanish. We integrate by parts on the second term

$$\int_0^T \frac{\partial \mathcal{L}}{\partial \dot{x}}\dot{\varepsilon}(t)dt = \left[\frac{\partial \mathcal{L}}{\partial \dot{x}}\varepsilon(t)\right]_0^T - \int_0^T \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}}\varepsilon(t)dt,$$

and use the fact that $\varepsilon(T) = \varepsilon(0) = 0$ to give us

$$\int_0^T \frac{\partial \mathcal{L}}{\partial \dot{x}} \dot{\varepsilon}(t) dt = \int_0^T \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{x}} \varepsilon(t) dt.$$

Putting this back together, we have

$$S[\tilde{x}(t) + \varepsilon(t)] = S[\tilde{x}(t)] + \int_0^T \left[\frac{\partial \mathcal{L}}{\partial x} - \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{x}} \right] \varepsilon(t) dt.$$

We need to make this second term vanish for any $\varepsilon(t)$, and this inherently requires

$$\frac{\partial \mathcal{L}}{\partial x} - \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{x}} = 0. \tag{2.8}$$

This is the Euler-Lagrange equation. This generalizes to the case where we have motion in three dimensions, meaning when we have an action involving functions $x_i(t)$, the Euler-Lagrange equation becomes three separate equations each for a different coordinate [7].

2.2.2 Newton's Second Law

In Newtonian mechanics, the correct Lagrangian is of the form $\mathcal{L} = T - U$, where T is the kinetic energy, and U is the potential energy. This can be proven with a Lagrangian in the following form:

$$\mathcal{L}(x_i, \dot{x}_i, t) = \frac{1}{2} m \sum_{i=1}^{3} \dot{x}_i^2 - U(x_i, t), \tag{2.9}$$

with this, we can find that

$$\frac{\partial \mathcal{L}}{\partial x_i} = -\frac{\partial U(x_j, t)}{\partial x_i} = F_i(x_j, t), \qquad \text{and} \qquad \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{x}_i} = m\ddot{x}_i,$$

which when plugged into equation 2.8 will give us

$$F_i(x_j, t) = m\ddot{x}_i,$$

which is Newton's second law [6].

2.2.3 Newtonian Gravity

This section illustrates the invariance of the action for Newtonian gravity under spatial rotations. It is necessary for the action to remain invariant under such rotations because it ensures that the resulting Euler-Lagrange equations also possess this property.

We will end up taking advantage of the fact that under rotations, a tensor will be invariant. This can be shown to hold true with Newton's second law, and it can also be shown for two matrices interacting via gravity. If we had a single point mass with a mass m at location x orbiting around another object with a fixed mass M at the origin, their corresponding kinetic and potential energies are the following:

$$T = \frac{1}{2} m \underbrace{\dot{x}^2}_{\text{invariant}}, \qquad U = -\frac{GmM}{\sqrt{\underbrace{x^2}_{\text{invariant}}}}. \tag{2.10}$$

From this can say that the Lagrangian is invariant under spatial rotations, and so is S, extremizing S guarantees that we will get sensible equations of motion [6].

Chapter 3

Theory: Special Relativity

In this chapter, we introduce the theoretical aspects necessary for understanding special relativity. Here, we will build upon the ideas presented in the previous chapter as well as lay the foundation for deriving the orbit under special relativity.

3.1 Invariance under Lorentz Transformations

We will now be shifting from Newtonian physics to special relativity. This means we will now use Lorentz transformations instead of rotational ones. The Lorentz transformations are a set of mathematical equations that relate the coordinates and time measurements between two inertial reference frames moving relative to each other at constant velocities.

For our situation, this will involve the use of four-vectors, with coordinates $x^\mu,\ \mu = 0, 1, 2, 3$. This now involves a new term, $x^0 = ct$, which is the time coordinate rescaled to have the same units as the other spatial coordinates (x^1, x^2, x^3).

In figure 3.1, we have two different frames, the one seen by the satellite, unprimed, and the one seen by the rocket, primed. In the unprimed frame, the satellite sees the rocket move away with velocity $\mathfrak{v}$, and the opposite is true for the primed frame, the rocket sees the satellite move away with the same velocity $\mathfrak{v}$ but this time in the opposite direction. In this situation, both $x'^2 = x^2$ and $x'^3 = x^3$, since the objects have no velocity component in those directions. In the following directions, the objects do have velocity components:

$$x'^0 = \frac{x^0}{\sqrt{1 - \frac{\mathfrak{v}^2}{c^2}}} - \frac{\frac{\mathfrak{v}}{c}x^1}{\sqrt{1 - \frac{\mathfrak{v}^2}{c^2}}}, \qquad x'^1 = \frac{x^1}{\sqrt{1 - \frac{\mathfrak{v}^2}{c^2}}} - \frac{\frac{\mathfrak{v}}{c}x^0}{\sqrt{1 - \frac{\mathfrak{v}^2}{c^2}}}. \tag{3.1}$$

A good check for these equations would be to check what x'^0 and x'^1 are for the non-relativistic limit ($\mathfrak{v} \ll c$), we see that

$$\begin{aligned} &x'^0 \approx x^0 - \frac{\mathfrak{v}}{c}x',\ \text{ or } t' \approx t - \frac{\mathfrak{v}}{c^2}x',\ \text{ but, } \mathfrak{v} \ll c,\ \text{ so } t' \approx t, \\ &x'^1 = x' - \frac{\mathfrak{v}}{c}x^0,\ \text{ or } x' = x - \mathfrak{v}t. \end{aligned} \tag{3.2}$$

We can see that this check returns some facts that we are familiar with, $t' \approx t$, time can be thought of as invariant when the transformation involves a speed $\mathfrak{v}$ which is non-relativistic. Also, $x' = x - \mathfrak{v}t$, the position

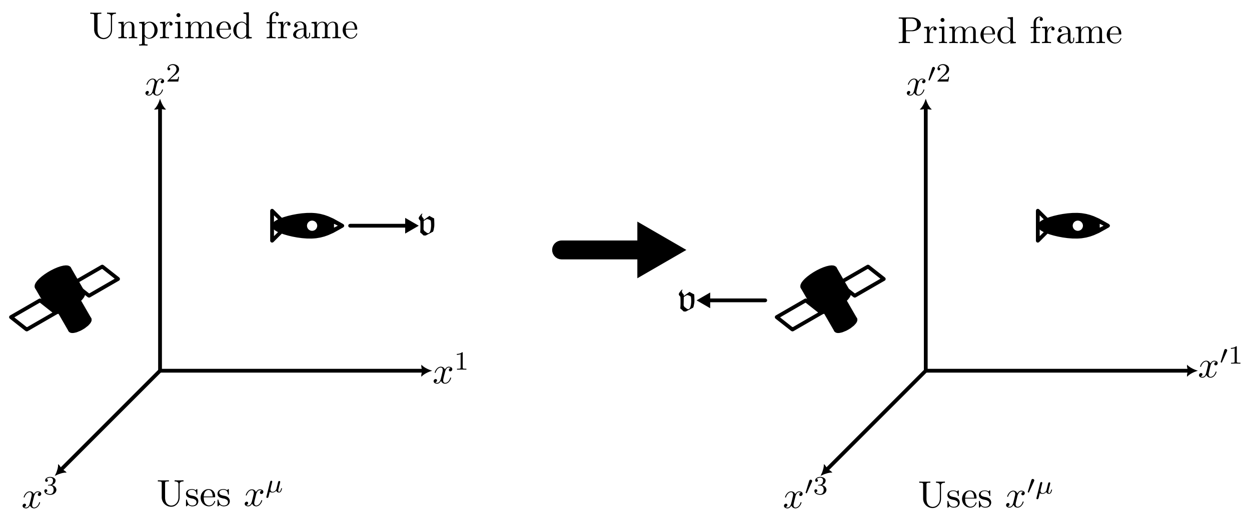


Figure 3.1: Diagram of different frames, the unprimed and primed frame.

of the satellite will be moved back by $-\mathfrak{v}t$ for the unprimed frame. Just as was the case for spatial rotations, we can also represent equations 3.1 in matrix notation and index notation:

$$\begin{bmatrix} x'^0 \\ x'^1 \\ x'^2 \\ x'^3 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{1-\frac{\mathfrak{v}^2}{c^2}}} & -\frac{\frac{\mathfrak{v}}{c}}{\sqrt{1-\frac{\mathfrak{v}^2}{c^2}}} & 0 & 0 \\ -\frac{\frac{\mathfrak{v}}{c}}{\sqrt{1-\frac{\mathfrak{v}^2}{c^2}}} & \frac{1}{\sqrt{1-\frac{\mathfrak{v}^2}{c^2}}} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{bmatrix}, \tag{3.3}$$

or

$$x'^\mu = \sum_{\nu=0}^{3} \Lambda^\mu_\nu x^\nu = \Lambda^\mu_\nu x^\nu. \tag{3.4}$$

In the index form, we can see that in special relativity, a repeated index where one index is “upper” and one “lower” always implies a sum over that index.

Now, we extend our approach to ensure the invariance of physics under Lorentz transformations, using four-vectors and tensors instead of just spatial vectors. To achieve this, we introduce a space-time generalization of the dot product, which must remain unchanged under our Lorentz transformations.

$$\eta_{\mu\nu} v^\mu w^\nu = \underbrace{-v^0 w^0 + v^1 w^1 + v^2 w^2 + v^3 w^3}_{\text{These should be invariant under Lorentz transformations}}. \tag{3.5}$$

Where $\boldsymbol{\eta}$ is the Minkowski metric, given by

$$\boldsymbol{\eta} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{3.6}$$

Proving the right side of the equation is invariant requires proving the following

$$-v'^0w'^0 + v'^1w'^1 + v'^2w'^2 + v'^3w'^3 = -v^0w^0 + v^1w^1 + v^2w^2 + v^3w^3. \tag{3.7}$$

We start by pointing out that $v^{2,3} = v'^{2,3}$ and similarly for $w^{2,3}$, such that we can ignore these components. Then we can transform the values of v'^0, v'^1 and w'^0, w'^1 in terms of v^0, v^1 and w^0, w^1, using equation 3.3. We define

$$\gamma = \frac{1}{\sqrt{1 - \frac{\mathfrak{v}^2}{c^2}}},$$

and as a result, our expression will be

$$-v^0w^0 + v^1w^1 = -\left(\gamma v^0 - \frac{\mathfrak{v}}{c}\gamma v^1\right)\left(\gamma w^0 - \frac{\mathfrak{v}}{c}\gamma w^1\right) + \left(\gamma v^1 - \frac{\mathfrak{v}}{c}\gamma v^0\right)\left(\gamma w^1 - \frac{\mathfrak{v}}{c}\gamma w^0\right),$$

which can be verified through algebraic calculations.

This demonstrates the invariance of the newly introduced special relativistic dot product under Lorentz transformations. Just as expected with the spatial transformations, we expect the physical laws to exhibit invariance under Lorentz transformations. To achieve this, we construct an action that maintains invariance, requiring the use of objects that transform appropriately, such as scalars, four-vectors, and more general four-tensors [8].

3.2 Four-velocity and Four-Momentum

To construct an action and comprehend physical dynamics, we require four-vector counterparts of velocity and momentum.

3.2.1 Velocity Transformation Rules

Before we construct these four-vector versions of velocity and momentum, we want to understand how the normal velocity vector will transform under Lorentz transformations.

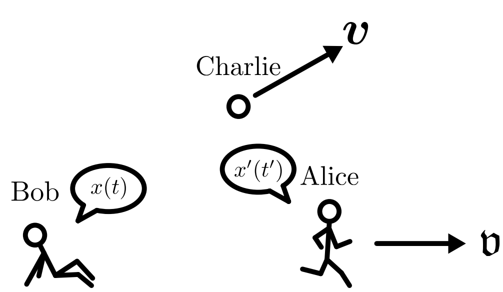


Figure 3.2: A graphic showing three different reference frames. Bob, who is stationary, Charlie, the reference frame of the ball with velocity $\boldsymbol{v}$, and Alice, who has a velocity of $\mathfrak{v}$. Bob reports Charlie's position as $x(t)$, and Alice reports Charlie's position as $x'(t')$.

In figure 3.2, we have a ball with some velocity $\boldsymbol{v}$, and three different reference frames, a stationary one, Bob, the reference frame of the ball itself, Charlie, and another that has some velocity $\mathfrak{v}$, Alice. Here, x is

the position of Charlie according to Bob, and x' is the x-component of the velocity of Charlie according to Alice. We will try to get back the components of velocity in the primed reference frame. We will now be finding the x-component of the velocity of Charlie according to Bob, and the x-component of the velocity of Charlie according to Alice. Which are given by

$$v_x = \frac{dx}{dt}, \qquad \text{and} \qquad v'_x = \frac{dx'}{dt'}, \tag{3.8}$$

which after plugging in the values for x' and t' from equation 3.1, gives us

$$v'_x = \frac{dx'}{dt'} = \frac{d\left(\gamma x - \gamma \mathfrak{v} t\right)}{d\left(\gamma t - \gamma \frac{\mathfrak{v}}{c^2} x\right)} = \frac{\frac{dx}{dt} - \mathfrak{v}}{1 - \frac{\mathfrak{v}}{c^2}\frac{dx}{dt}} = \frac{v_x - \mathfrak{v}}{1 - \frac{\mathfrak{v} v_x}{c^2}}. \tag{3.9}$$

For v'_y, will have

$$v'_y = \frac{dy'}{dt'} = \frac{dy}{d\left(\gamma t - \gamma \frac{\mathfrak{v}}{c^2} x\right)} = \frac{\frac{dy}{dt}}{\gamma\left(1 - \frac{\mathfrak{v}}{c^2}\frac{dx}{dt}\right)} = \frac{v_y}{\gamma\left(1 - \frac{\mathfrak{v} v_x}{c}\right)}, \tag{3.10}$$

the answer is similar for v'_z. In total we have

$$v'_x = \frac{v_x - \mathfrak{v}}{1 - \frac{\mathfrak{v} v_x}{c^2}}, \qquad v'_y = \frac{v_y}{\gamma\left(1 - \frac{\mathfrak{v} v_x}{c}\right)}, \qquad v'_z = \frac{v_z}{\gamma\left(1 - \frac{\mathfrak{v} v_x}{c}\right)}. \tag{3.11}$$

We can see that for the limit $v_x \ll c$, we have the normal rules that you would expect to see in everyday life. The ball's x component of velocity is subtracted from Alice's since Alice had velocity in the x direction. All of the other components stay the same.

However, when relativistic speeds get considered, even the components perpendicular to the direction Alice is moving transform, due to the effect of time dilation. Because of this extra factor, these equations don't transform nicely, or in other words, they are not components of a four-vector. This requires a step back to rethink things before we can get invariant equations for the primed velocity components.

3.2.2 Defining Four-Velocity and Four-Momentum

We will instead use the proper time "τ" which is the time elapsed in the frame of the object moving. Consider two space-time events separated by small amounts dx^μ, from this, we can define the separation $d\tau$ to be

$$\begin{aligned} c^2 d\tau^2 &= (dx^0)^2 - (dx^1)^2 - (dx^2)^2 - (dx^3)^2, \\ &= -\eta_{\mu\nu} dx^\mu dx^\nu. \end{aligned} \tag{3.12}$$

where now $d\tau$ is the value for the time separation in the frame of the moving object (where $dx^i = 0$). But also, $d\tau$ is invariant under Lorentz transformations, because it is in the form of the new, four-vector version of the dot product.

The following object

$$v^\mu = \frac{dx^\mu}{d\tau},$$

is now a four-vector, known as the four-velocity. The point here is that this fixes the issue that arose in the normal velocity transformation rules because the denominator in the derivative no longer transforms and the numerator does transform as a four-vector.

For Bob, we will have

$$c^2 d\tau^2 = (dx^0)^2 \left(1 - \left(\frac{dx^1}{dx^0}\right)^2 - \left(\frac{dx^2}{dx^0}\right)^2 - \left(\frac{dx^3}{dx^0}\right)^2\right), \tag{3.13}$$

which can be simplified to

$$c^2 d\tau^2 = c^2 dt^2 \left(1 - \frac{v^2}{c^2}\right),$$

where we notice that the final v^2 is the normal, three-dimensional velocity squared of the object. After rearranging and simplification it becomes

$$v^0 = \frac{dx^0}{d\tau} = \frac{c}{\sqrt{1 - \frac{v^2}{c^2}}}, \tag{3.14}$$

Now we have v^0, to get the rest we employ a trick with the derivatives:

$$v^1 = \frac{dx^1}{d\tau} = \frac{c}{\sqrt{1 - \frac{v^2}{c^2}}} \frac{dx^1}{dx^0} = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} v^x. \tag{3.15}$$

Te process is similar for the other two vector components, in the end, we have

$$v^0 = \frac{c}{\sqrt{1 - \frac{v^2}{c^2}}}, \qquad v^1 = \frac{v_x}{\sqrt{1 - \frac{v^2}{c^2}}}, \qquad v^2 = \frac{v_y}{\sqrt{1 - \frac{v^2}{c^2}}}, \qquad v^3 = \frac{v_z}{\sqrt{1 - \frac{v^2}{c^2}}}. \tag{3.16}$$

This is Charlie's four-velocity according to Bob, and it transforms the way a four-vector does. We have gone from the Galilean velocities, (v_x, v_y, v_z), and now moved to a new relativistic velocity four-vector. Here, v^2 and v_x are the traditional, Galilean objects.

We can define a new quantity, the four-momentum, which is related to the four-velocity. The four-momentum is defined as $p^\mu = mv^\mu$, which gives

$$p^0 = \frac{mc}{\sqrt{1 - \frac{v^2}{c^2}}}, \qquad p^1 = \frac{mv_x}{\sqrt{1 - \frac{v^2}{c^2}}}, \qquad p^2 = \frac{mv_y}{\sqrt{1 - \frac{v^2}{c^2}}}, \qquad p^3 = \frac{mv_z}{\sqrt{1 - \frac{v^2}{c^2}}}. \tag{3.17}$$

We note that cp^0 has the approximation

$$cp^0 = \frac{mc^2}{\sqrt{1 - \frac{v^2}{c^2}}} \approx mc^2 + \frac{1}{2} mv^2 + ...$$

by using a Taylor expansion at the limit $v \ll c$. From this, we can see that it is essentially the relativistic energy, including both kinetic energy and mass energy terms.

We have

$$E = cp^0 = \frac{mc^2}{\sqrt{1 - \frac{v^2}{c^2}}}, \tag{3.18}$$

and we say the spatial components of the four-vector form the relativistic momentum vector $\boldsymbol{p}$.

If we use our Lorentz-invariant version of the dot product, we can derive an interesting fact:

$$\eta_{\mu\nu}p^{\mu}p^{\nu} = -\left(p^0\right)^2 + \left(p^1\right)^2 + \left(p^2\right)^2 + \left(p^3\right)^2,$$

using equation 3.12. With the use of equations 3.17 it becomes

$$-\left(p^0\right)^2 + \left(p^1\right)^2 + \left(p^2\right)^2 + \left(p^3\right)^2 = m^2c^2.$$

With our knowledge of the four-momentum in mind, we can say this equation becomes

$$E^2 = m^2c^4 + ||\boldsymbol{p}||^2c^2. \tag{3.19}$$

This equation gives us a relativistic relationship between energy, momentum, and mass [8].

3.3 Relativistic Dynamics

Our basic method is to start with a statement of relativistic energy conservation and deduce from that what the relativistic correction to Newton's second law should be. From there we will move on to write the relativistic action which has this equation as its Euler-Lagrange equation.

We are going to temporarily assume that we are working in one spatial dimension and one-time dimension. Then, the total energy in a system with some potential energy function $U(x)$ should be a sum of this potential energy with the object cp^0 defined earlier, which includes both the mass and kinetic energy. Our expression should therefore be

$$E = \frac{mc^2}{\sqrt{1-\frac{v^2}{c^2}}} + U(x). \tag{3.20}$$

This value should be conserved, meaning $\frac{dE}{dt} = 0$. When performing the derivative, we see that

$$0 = \frac{d}{dt}\left[\frac{mc^2}{\sqrt{1-\frac{v^2}{c^2}}} + U(x)\right] = mc^2\frac{d}{dv}\left(\frac{1}{\sqrt{1-\frac{v^2}{c^2}}}\right)\frac{dv}{dt} + \frac{dU}{dx}\frac{dx}{dt} = \frac{mav}{\left(1-\frac{v^2}{c^2}\right)^{3/2}} - vF(x),$$

where the force $F(x)$ should be $F(x) = -\frac{dU}{dx}$. Upon solving for $-\frac{dU}{dx}$ we get the equation

$$F(x) = \frac{ma}{\left(1-\frac{v^2}{c^2}\right)^{3/2}}. \tag{3.21}$$

This is Newton's second law when including special relativity. This can also be written as

$$F(x) = \frac{d}{dt}\left(\frac{mv}{\sqrt{1-\frac{v^2}{c^2}}}\right) = \frac{dp^1}{dt}, \tag{3.22}$$

where p^1 is the spatial component of the relativistic momentum. When going back to working in three spatial dimensions we get

$$\frac{dp^i}{dt} = F^i = -\frac{\partial U}{\partial x^i}. \tag{3.23}$$

We will now show that $\frac{dp_i}{dt} = F_i$, with $p_i = \frac{mv_i}{\sqrt{1-\frac{v^2}{c^2}}}$ is the Euler-Lagrange equation for

$$S = \int_0^T \left(mc^2 \sqrt{1 - \frac{v^2}{c^2}} - U(x^\mu) \right) dt. \tag{3.24}$$

Our Lagrangian then would be

$$\mathcal{L} = mc^2 \sqrt{1 - \sum_{i=1}^{3} \frac{\dot{x}_i^2}{c^2}} - U(x^\mu),$$

and so the following must be true

$$\frac{\partial \mathcal{L}}{\partial x_i} = -\frac{\partial U(x_j, t)}{\partial x_i} = F_i(x_j, t). \tag{3.25}$$

Plugging in the values for $F_i(x_j, t)$, we will have

$$\frac{\partial \mathcal{L}}{\partial \dot{x}_i} = mc^2 \frac{\partial}{\partial \dot{x}_i} \left(1 - \sum_{i=1}^{3} \frac{\dot{x}_j^2}{c^2} \right)^{1/2} = -\frac{m\dot{x}_i}{\sqrt{1 - \frac{v^2}{c^2}}} = -p^i.$$

This means we can say that

$$\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{x}_i} = -\frac{dp^i}{dt}. \tag{3.26}$$

These equations can be plugged into the Euler-Lagrange equation,

$$\frac{\partial \mathcal{L}}{\partial x_i} - \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{x}_i} = 0, \tag{3.27}$$

which gives us

$$F_i(x_j, t) = \frac{dp^i}{dt}, \tag{3.28}$$

giving us that the derivative of momentum is the force [9].

3.3.1 Invariance of the Action Under Lorentz Transformations

We want to make sure the action is invariant under Lorentz transformations, as it will let us combine Newtonian gravity with special relativity, giving us another view of gravity. We will start with rewriting the action from equation 3.24 as two terms,

$$S = \int mc^2 \sqrt{1 - \frac{v^2}{c^2}}\, dt \; - \int U(x_i)\, dt = mc^2 \int d\tau \; - \int U(x^\mu)\, dt.$$

Here we observe that the integrand in the first term is responsible for outputting the proper time. This means that this first term is naturally Lorentz invariant and that in the absence of any forces, the natural interpretation of the action principle in special relativity is that the proper time elapsed for the particle is extremized.

It is possible to define a potential that is Lorentz invariant, but the potential for Newtonian gravity will not be. For gravity,

$$U(\boldsymbol{r}) = \frac{-GmM}{\sqrt{{x_1}^2 + {y_1}^2 + {z_1}^2}},$$

here, this potential is invariant under spatial rotations, but not under Lorentz transformations. This is because the denominator of the potential is the length of a spatial vector, not the magnitude of a space-time four-vector. This indicates that there is no truly consistent way to combine Newtonian gravity with special relativity. The only method we can use is general relativity. However, we will still try to find the orbital shape by combining Newtonian gravity with special relativity. This is an ad hoc construction that ignores this issue, while not fully accurate, it can and does give useful insights [9].

Chapter 4

Theory: General Relativity

We want to generalize the idea of flat space-time and Lorentz transformations to a discussion of a curved space-time. This means we will define the metric and discuss generalized coordinate transformations. Then, we want to discuss the action of a particle in curved space-time. This will replace the action from equation 3.24. Most significantly, this action does not have a separate potential energy term in it. Instead, it incorporates the effect of gravity into the curvature of space-time. This action is the action for a point particle in a curved background. Next, we want to discuss the action for the space-time itself. This is the action for the curved background itself, and it involves defining Christoffel symbols, covariant derivatives, the Reimann tensor, the Ricci tensor, and the Ricci scalar, before finally writing down the action and then extremizing it. Lastly, we want to apply this action such that we can understand what space-time would result from having a single, massive point source M at the origin. This chapter uses Sean Carroll's *Spacetime and Geometry: An Introduction to General Relativity* [4].

4.1 The Metric and Generalized Coordinate Transformations

We are starting by generalizing what we did for Euclidian space and then for space-time, this is to introduce material related to the geometry of the space-time we are working in.

4.1.1 The Metric

The metric is the mathematical description of a curved space-time, and we are going to define it first by introducing a series of examples. This will give us a better intuition about what the metric actually means.

Euclidian Flat 2D Space

For flat, two-dimensional space, if we have a point located at (x, y) and another at $(x + dx, y + dy)$ we understand that the distance between these two points must be

$$ds^2 = dx^2 + dy^2.$$

This familiar answer is simply a result of using the Pythagorean theorem.

Polar Flat 2D Space

We can also express flat, two-dimensional space using polar coordinates and perform the same problem. This time we will have a point at (r, θ) and one at $(r + dr, \theta + d\theta)$, and their distance will be

$$ds^2 = dr^2 + r^2 d\theta^2.$$

This answer is less familiar but still understandable, especially considering the rule that $dA = r dr d\theta$. The area infinitesimal $dA = r dr\, d\theta$ comes from multiplying a displacement in the radial direction dr by a displacement in the perpendicular angular direction $r d\theta$, to get the area of a small rectangle. To get the distance between two points, you would want to use those two displacements as the sides of a right triangle for which you wanted the hypotenuse, instead.

2D Surface of a Sphere

We will now transition to where our space is the two-dimensional surface of a sphere with a radius R, and play the same game. We have two points (θ, ϕ), and $(\theta + d\theta, \phi + d\phi)$, and the distance between them is

$$ds^2 = \left(d\theta^2 + \sin^2\theta d\phi^2\right) R^2.$$

Once again this answer may look different than one would expect, however, the area of a patch on a sphere is $R^2 \sin\theta\, d\theta\, d\phi$ because we are multiplying together two displacements $R d\theta$ and $R \sin\theta d\phi$. To get the distance between the points, we treat the displacements as the sides of a right triangle and find the hypotenuse. This only works because the points are close, such that we can estimate the arc lengths as a straight distance.

3+1 Space-Time

Now we will incorporate time as well as space, so we will have a point x^μ and a nearby point $x^\mu + dx^\mu$. The space-time distance between these two points will be

$$ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu = -(dx^0)^2 + (dx^1)^2 + (dx^2)^2 + (dx^3)^2,$$

where $\eta_{\mu\nu}$ comes from equation 3.6. We learned from the previous chapter that ds^2 is Lorentz invariant because it is in the form of a dot product. If we assume the two points were "located" at the same time then we would just have the Euclidian flat three-dimensional story.

The General Metric

We see that there is a relationship between all of these examples of the distance between two points for different space-times or coordinates. If we generalize each example and assume we are in a space-time that is curved, we can define $g_{\mu\nu}$ such that

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu. \tag{4.1}$$

4.1.2 Coordinate transforms

It is feasible to use various coordinate systems to depict the same space or space-time, similar to the case of Cartesian and two-dimensional polar coordinates. The selection of coordinates should not have any impact on the fundamental physical laws, similar to how the choice of reference frame or axis orientation did not affect our previous discussions.

We start by assuming we have two different coordinates x^μ and x'^μ, and we give the general rule that the object ds^2 should be the same in any coordinates, so that

$$ds^2 = g_{\mu\nu}(x^\rho) dx^\mu dx^\nu = g'_{\mu\nu}(x'^\rho) dx'^\mu dx'^\nu .$$

From which follows the transformation rule for the metric

$$g'_{\mu\nu} = g_{\rho\sigma} \frac{\partial x^\rho}{\partial x'^\mu} \frac{\partial x^\sigma}{\partial x'^\nu} . \tag{4.2}$$

If we use $g_{\mu\nu}$ in the creation of our new "dot product", then we expand our vectors in terms of unit vectors as

$$\boldsymbol{v} = v^\mu \boldsymbol{e}_\mu ,$$

where $\boldsymbol{e}_\mu$ are our unit vectors. Then the new "dot product" will be

$$\boldsymbol{v} \cdot \boldsymbol{w} = g_{\mu\nu} v^\mu v^\nu ,$$

and simply by inserting two different expanded vectors, we can find that

$$g_{\mu\nu} = \boldsymbol{e}_\mu \cdot \boldsymbol{e}_\nu .$$

This, combined with the transformation rule for the metric, implies transformation rules for the unit vectors will be given as

$$\boldsymbol{e}'_\mu = \boldsymbol{e}_\rho \frac{\partial x^\rho}{\partial x'^\mu} . \tag{4.3}$$

Finally, if a vector itself ($\boldsymbol{v}$) doesn't actually transform when we change coordinates, only its components change. Using $\mathbf{v} = v^\mu \mathbf{e}_\mu$, we can then deduce the transformation rules for the components, which are

$$v'^\mu = v^\rho \frac{\partial x'^\mu}{\partial x^\rho} .$$

A Worked Example

In this subsection, we will build intuition that equation 4.2 and equation 4.3 work as intended. Starting from equation 4.2, we will assume our unprimed coordinates are the Cartesian coordinates, x, y, and our primed coordinates are the polar coordinates, r, θ. Then we will start with the metric for flat two-dimensional space:

$$\boldsymbol{g} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} ,$$

and we will show

$$\boldsymbol{g}' = \begin{bmatrix} 1 & 0 \\ 0 & r^2 \end{bmatrix} .$$

Now with $x = r\cos\theta$ and $y = r\sin\theta$ it can be shown that

$$g_{rr} = g_{xx} \frac{\partial x}{\partial r}\frac{\partial x}{\partial r} + g_{xy} \frac{\partial x}{\partial r}\frac{\partial y}{\partial r} + g_{yx} \frac{\partial y}{\partial r}\frac{\partial x}{\partial r} + g_{yy} \frac{\partial y}{\partial r}\frac{\partial y}{\partial r} = \cos^2\theta + \sin^2\theta = 1,$$

and

$$g_{\theta\theta} = g_{xx} \frac{\partial x}{\partial \theta}\frac{\partial x}{\partial \theta} + g_{xy} \frac{\partial x}{\partial \theta}\frac{\partial y}{\partial \theta} + g_{yx} \frac{\partial y}{\partial \theta}\frac{\partial x}{\partial \theta} + g_{yy} \frac{\partial y}{\partial \theta}\frac{\partial y}{\partial \theta} = \left(\cos^2\theta + \sin^2\theta\right) r^2 = r^2 .$$

We can also show that

$$g_{r\theta} = g_{\theta r} = g_{xx}\frac{\partial x}{\partial \theta}\frac{\partial x}{\partial r} + g_{xy}\frac{\partial x}{\partial \theta}\frac{\partial y}{\partial r} + g_{yx}\frac{\partial y}{\partial \theta}\frac{\partial x}{\partial r} + g_{yy}\frac{\partial y}{\partial \theta}\frac{\partial y}{\partial r} = -r\sin\theta\cos\theta + r\sin\theta\cos\theta = 0.$$

This means that the metric $\boldsymbol{g}'$ is correct, and furthermore, equation 4.2 works correctly for going from Cartesian coordinates to polar coordinates.

For equation 4.3, we will assume our unprimed coordinates and primed coordinates are the same as in the previous example. With this in place, it can be shown that

$$\hat{r} = \hat{x}\frac{\partial x}{\partial r} + \hat{y}\frac{\partial y}{\partial r} = \cos\theta\hat{x} + \sin\theta\hat{y},$$

and

$$\hat{\theta} = \hat{x}\frac{\partial x}{\partial \theta} + \hat{y}\frac{\partial y}{\partial \theta} = -r\sin\theta\hat{x} + r\cos\theta\hat{y}.$$

These expressions should be familiar from discussions of how to describe classical mechanics in different coordinates.

4.2 The Path of a Particle in a Curved Space-Time

In special relativity, we had an action that was in the form $S = \int d\tau - \int U(x^\mu)dt$, where the information about the forces on the particle is in the potential term. In general relativity, we eliminate the potential term and instead, we just have $S = \int d\tau$. This time we use the $d\tau$ term as defined with a curved space-time metric. This is the path the particle follows, which is simply the path that extremizes the proper time of the particle in the curved background. This path is also known as a "geodesic." Our action is now

$$S = \int \sqrt{-g_{\mu\nu}dx^\mu dx^\nu}. \tag{4.4}$$

4.2.1 Great Circles on a Sphere: An Example

To try to get a sense of what this action extremization means, we will be applying it to the case of a two-dimensional sphere geometry. Instead of extremizing the proper time, we're simply extremizing the distance between two points. The result of which should be a "great circle", a circle with radius R on the sphere, which is the same path an airplane travels.

In this case, the shortest distance between two points on a sphere would follow from extremizing

$$\begin{aligned} S &= \int \sqrt{d\theta^2 + d\phi^2\sin^2\theta}, \\ S[\phi(\theta)] &= \int_{\theta_0}^{\theta_f} \sqrt{1 + \frac{d\phi^2}{d\theta^2}\sin^2\theta}\, d\theta. \end{aligned} \tag{4.5}$$

In order to solve the Euler-Lagrange equation, we will need to solve

$$\frac{d}{d\theta}\frac{\partial \mathcal{L}}{\partial \phi'} - \frac{\partial \mathcal{L}}{\partial \phi} = 0, \qquad \text{with} \qquad \mathcal{L} = \sqrt{1 + \frac{d\phi^2}{d\theta^2}\sin^2\theta},$$

we can see that the $\frac{\partial \mathcal{L}}{\partial \phi}$ term will be zero, and then we'll be left with

$$\frac{\partial \mathcal{L}}{\partial \phi'} = C,$$

where C is some constant. We will have

$$\frac{\partial \mathcal{L}}{\partial \phi'} = \frac{\sin^2 \theta \phi'}{\sqrt{1 + \sin^2 \theta \phi'^2}} = C.$$

We can then simplify this equation and solve for ϕ', which gives us

$$\phi' = \frac{C}{\sin \theta \sqrt{\sin^2 \theta - C^2}}.$$

We then have to solve the equation for ϕ, which is done by integrating both sides. Here we can see that C is required to be less than one, or else the answer would be imaginary. Integrating gives us

$$\cos(\phi - \phi_0) = (\beta \cot \theta), \tag{4.6}$$

where $\beta = \frac{C}{\sqrt{1-C^2}}$.

We can check our answer by finding the equation for a great circle, which should form from the intersection of the sphere with a plane that passes through the origin. Such a plane would be defined by

$$ax + by + cz = 0,$$

we will then make the equation in spherical coordinates:

$$a \cos \phi \sin \theta + b \sin \phi \sin \theta + c \cos \theta = 0,$$

and then rearrange to get

$$a \cos \phi + b \sin \phi = -c \cot \theta,$$

the left side of the equation can be changed using trigonometric identities:

$$\sqrt{a^2 + b^2} \cos(\phi - \phi_0) = -c \cot \theta, \qquad \text{where} \qquad \theta_0 = \arctan b/a.$$

This becomes

$$(\phi - \phi_0) = \arccos(\beta \cot \theta), \qquad \text{where} \qquad \beta = -\frac{c}{\sqrt{a^2 + b^2}}. \tag{4.7}$$

We see that in this case, we recover the equation for a great circle by extremizing our action – finding the shortest path on the surface of a sphere. However, what we *really* want to do is find the path of a particle in the curved space-time. Where the curvature is created by a massive source M at the origin. This is done in a similar manner, but we must first discuss what the curved space-time actually is.

4.3 The Action for the Metric and Einstein's Equations

The action for a particle's path in a curved background is not the only action involved in general relativity. We will also have an action that determines what the shape of the background is, given a massive source. The Euler-Lagrange equations for that action are "Einstein field equations," and by solving them we determine what metric to use in order to model the particle's motion around a single massive source. In order to construct this action, and Einstein's equations from it, we must further develop the technology of general relativity. This means we will introduce covariant derivatives, Christoffel symbols, and the Reimann and Ricci curvature tensors.

4.3.1 Covariant Derivatives and Christoffel Symbols

We will start with the idea of a vector field ($\boldsymbol{v} = v^\mu \boldsymbol{e}_\mu$), which is a function of the coordinates $\boldsymbol{v}(x^\mu)$. We want to make a derivative of this vector with respect to one of the coordinates, however, we face an issue where the unit vectors also depend on the coordinates. We get

$$\frac{\partial \boldsymbol{v}}{\partial x^\nu} = \frac{\partial v^\mu}{\partial x^\nu}\boldsymbol{e}_\mu + v^\mu \frac{\partial \boldsymbol{e}_\mu}{\partial x^\nu}.$$

However, the object $\frac{\partial \boldsymbol{v}}{\partial x^\mu}$ must itself be something that can be expanded in terms of the unit vectors. This implies that we must be able to define objects $\Gamma^\rho_{\mu\nu}$ so that

$$\frac{\partial \boldsymbol{e}_\mu}{\partial x^\nu} = \Gamma^\rho_{\mu\nu}\boldsymbol{e}_\rho, \tag{4.8}$$

these objects are called Christoffel symbols. Putting this together, we can now say

$$\frac{\partial \boldsymbol{v}}{\partial x^\nu} = \Big(\partial_\mu v^\rho + \Gamma^\rho_{\mu\nu} v^\nu\Big)\boldsymbol{e}_\rho.$$

Finally, we note that this object $\frac{\partial \boldsymbol{v}}{\partial x^\mu}$ must transform nicely, which in this case means like an object with one lower index. Because of this and the fact that the $\boldsymbol{e}_\rho$ term also transformed nicely, the whole object $\partial_\mu v^\rho + \Gamma^\rho_{\mu\nu} v^\nu$ transforms nicely, this time how an object with one lower index and one upper index should. We define this as the covariant derivative,

$$\nabla_\mu v^\rho = \partial_\mu v^\rho + \Gamma^\rho_{\mu\nu} v^\nu. \tag{4.9}$$

We can also show the equation for $\nabla_\mu v_\nu$, to do this we first start with the definition:

$$\nabla_\mu \left(v_\nu w^\nu\right) = \partial_\mu \left(v_\mu w^\nu\right). \tag{4.10}$$

Using the product rule, simplification, and a change of indices, we can prove that

$$\nabla_\mu v_\nu = \partial_\mu v_\nu - v_\sigma \Gamma^\sigma_{\nu\mu}.$$

This generalizes to other index structures, for example for the Christoffel symbol we will have

$$\nabla_\mu \Gamma^\rho_{\sigma\tau} = \partial_\mu \Gamma^\rho_{\sigma\tau} + \Gamma^\alpha_{\mu\sigma}\Gamma^\rho_{\alpha\tau} - \Gamma^\alpha_{\mu\tau}\Gamma^\rho_{\alpha\sigma}.$$

We have defined the Christoffel symbol, however, we also need to know how to compute it from a generic metric. We require $\nabla_\mu \nabla_\nu \phi = \nabla_\nu \nabla_\mu \phi$, so that covariant partial derivatives commute when acting on scalars. This isn't always true in curved backgrounds, but it will be true for most cases of interest. A curved background for which it is true is known to be "torsion free". We can come to further conclusions when performing $\nabla_\mu \nabla_\nu \phi$ and $\nabla_\nu \nabla_\mu \phi$, where ϕ is a generic vector. For the first, we have:

$$\nabla_\mu \nabla_\nu \phi = \nabla_\mu \left(\partial_\nu \phi\right) = \partial_\mu \left(\partial_\nu \phi\right) - \Gamma^\sigma_{\nu\mu}\left(\partial_\sigma \phi\right),$$

and for the second, we have:

$$\nabla_\nu \nabla_\mu \phi = \nabla_\mu \left(\partial_\nu \phi\right) = \partial_\nu \left(\partial_\mu \phi\right) - \Gamma^\sigma_{\mu\nu}\left(\partial_\sigma \phi\right).$$

When the subscripts change order it changes the order of all of the subscripts on the other side of the equation. From now on we will have that $\Gamma^{\sigma}_{\mu\nu} = \Gamma^{\sigma}_{\nu\mu}$. With this in place, we will look at the following:

$$\partial_\mu g_{\nu\sigma} = \partial_\mu \left(\boldsymbol{e}_\nu \cdot \boldsymbol{e}_\sigma\right) = \partial_\mu \boldsymbol{e}_\nu \cdot \boldsymbol{e}_\sigma + \boldsymbol{e}_\nu \cdot \partial_\mu \boldsymbol{e}_\sigma = \Gamma^{\rho}_{\mu\nu} \boldsymbol{e}_\rho \cdot \boldsymbol{e}_\sigma + \boldsymbol{e}_\nu \cdot \Gamma^{\rho}_{\mu\sigma} \boldsymbol{e}_\rho \boldsymbol{e}_\rho,$$

where we used the product rule, and the definition of the Christoffel symbol from equation 4.8. From here we will plug these values into the following:

$$\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\sigma\mu} - \partial_\sigma g_{\mu\nu} = \Gamma^{\rho}_{\mu\nu} g_{\rho\sigma} + \Gamma^{\rho}_{\mu\sigma} g_{\nu\sigma} + \Gamma^{\rho}_{\nu\sigma} g_{\rho\mu} + \Gamma^{\rho}_{\mu\nu} g_{\rho\sigma} - \Gamma^{\rho}_{\mu\sigma} g_{\nu\rho} - \Gamma^{\rho}_{\nu\sigma} g_{\rho\mu}.$$

We can then cancel like terms,

$$2\Gamma^{\rho}_{\mu\nu} g_{\rho\sigma} = \left(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\sigma\mu} - \partial_\sigma g_{\mu\nu}\right),$$

and multiply both sides by $g^{\sigma\alpha}/2$, giving us

$$\Gamma^{\alpha}_{\mu\nu} = \frac{1}{2} g^{\sigma\alpha} \left(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\sigma\mu} - \partial_\sigma g_{\mu\nu}\right). \tag{4.11}$$

4.3.2 The Reimann Curvature Tensor

When looking at a metric, it is impossible to tell if it represents a genuinely curved space-time, or if it is flat, but expressed in awkward coordinates. Because of this we want to create an object that mathematically expresses this curvature, which is the Reimann tensor. Unlike the idea of a torsion-free space-time, it is impossible to have curvature and have covariant partial derivatives commute on a vector. In fact, we can say that the failure of those covariant partial derivatives to commute is the essence of curvature, so we define the Reimann curvature tensor to be the object $R^{\rho}_{\sigma\mu\nu}$ such that $\nabla_\mu \nabla_\nu v^\rho - \nabla_\nu \nabla_\mu v^\rho = R^{\rho}_{\sigma\mu\nu} v^\sigma$. By simply applying the rules of the covariant derivatives, we then come to the formula for the Reimann curvature tensor in terms of the Christoffel symbols:

$$R^{\beta}_{\rho\mu\nu} = \partial_\mu \Gamma^{\beta}_{\rho\nu} - \partial_\nu \Gamma^{\beta}_{\rho\mu} + \Gamma^{\alpha}_{\rho\nu} \Gamma^{\beta}_{\alpha\mu} - \Gamma^{\alpha}_{\rho\mu} \Gamma^{\beta}_{\alpha\nu}. \tag{4.12}$$

This curvature tensor would vanish if it was computed for two-dimensional flat space in polar coordinates, but not if it was computed for the two-dimensional surface of a sphere. This is because the two-dimensional surface of a sphere is actually a curved space. By simply applying the rules for the covariant derivatives, we then come to the formula for the Reimann curvature tensor in terms of the Christoffel symbols.

4.3.3 The Ricci Curvature Tensor and Scalar

Since the Reimann object is a tensor and transforms nicely, it can be used to construct two other objects which also transform nicely. These objects are the Ricci tensor, which transforms as a two-index object, and the Ricci scalar, which doesn't transform. In particular, the Ricci scalar represents a coordinate-free measure of the curvature of the space you are in. The Ricci curvature tensor is defined by

$$R_{\rho\nu} = R^{\beta}_{\rho\beta\nu}, \tag{4.13}$$

and for the Ricci scalar, we contract again giving us

$$g^{\rho\nu} R_{\rho\nu} = R. \tag{4.14}$$

It can also be shown that the Ricci tensor is symmetric ($R_{\rho\nu} = R_{\nu\rho}$). When computing the Ricci scalar for the surface of a sphere, we get that the Ricci scalar is $\frac{1}{R^2}$, where R is the radius of the sphere. This is because it is the same value everywhere on the sphere (spheres are equally as curved everywhere on the surface), and it is inversely related to the radius of the sphere. This means as the sphere gets bigger, the flatter the surface looks.

4.3.4 Finding the Action for 3+1 SpaceTime

First, we will construct an action that is invariant under coordinate transforms. Then we need to extremize it to find its Euler-Lagrange equations, which as we mentioned will be Einstein's equations.

In order to find the action, we need it to be invariant, however, the object we have to integrate over, $dx^0dx^1dx^2dx^3$, is not invariant. We need to use the object

$$\sqrt{-\det g}\, dx^0dx^1dx^2dx^3$$

which is invariant. It can be shown that this object reproduces familiar results when applied to Cartesian, polar, and spherical coordinates. This value comes into play whenever an integral is being performed over all coordinates in the coordinate system, a negative sign is multiplied under the square root when involving space-time.

A natural object to include in the action would be the Ricci scalar, which as we mentioned is itself invariant. We also need a term that includes the effects of mass/energy, which should act as a source for the curvature. This object is represented by $T_{\mu\nu}$, which must be contracted with the metric so that it is also invariant. This means that the integrand has to necessarily be $R + g^{\mu\nu}T_{\mu\nu}$.

For now, we will focus on the Ricci scalar (from equation 4.14). We start with

$$S[g_{\mu\nu}] = \int_{\mathbb{R}^{3+1}} \sqrt{-\det g}\, (g^{\mu\nu}R_{\mu\nu})\, d^{3+1}x, \tag{4.15}$$

which then becomes

$$S[g_{\mu\nu}] = \int_{\mathbb{R}^{3+1}} \sqrt{-\det g}\, (R + g^{\mu\nu}T_{\mu\nu})\, d^{3+1}x,$$

Since we know that the definition of the Ricci scalar depends on both first and second derivatives of the metric, our Lagrangian would be

$$\mathcal{L}\left(g_{\mu\nu}(x^\rho), \partial_\rho g_{\mu\nu}, \partial_\rho\partial_\sigma g_{\mu\nu}\right).$$

In order to find the Euler-Lagrange equations we want to extremize this action. This is similar to our original argument in Chapter 2.2 where we derived the Euler-Lagrange equations. In order to find the full expression for the action we need to add a small perturbation to the action, which would look like

$$S[g_{\mu\nu} + \delta g_{\mu\nu}] = S[g_{\mu\nu}] = \int_{\mathbb{R}^{3+1}} \underbrace{(\qquad\qquad)}_{=0\to\text{(E-L equations)}} \delta g_{\mu\nu} \qquad + \qquad \text{Higher order values.}$$

The center component's integrand would end up equaling zero, giving us the Euler-Lagrange equations.

The small perturbation component of equation 4.15 would therefore be

$$\delta S = \int_{\mathbb{R}^{3+1}} \left(\left(\delta\sqrt{-\det g}\right) g^{\mu\nu}R_{\mu\nu} + \sqrt{-\det g}\delta g^{\mu\nu}R_{\mu\nu} - \sqrt{-\det g}g^{\mu\nu}\delta R_{\mu\nu}\right) d^{3+1}x. \tag{4.16}$$

Now we need to work out the relationship between $\delta g_{\mu\nu}$ and the small perturbation of the inverse metric, $\delta g^{\mu\nu}$, and vice versa. For $\delta g_{\mu\nu}$, we can sum a matrix with another that is made up of small perturbations of the original matrix's components. This will create a matrix that has components that are the sum of the original matrix's components and small perturbations of the original matrix's components. In order to find the expression for $\delta g^{\mu\nu}$, consider a matrix $\mathbb{M}$, a small perturbation of it $\delta\mathbb{M}$, the matrix's inverse $\mathbb{M}^{-1}$, and the perturbation of that, $\delta(\mathbb{M}^{-1})$.

We can see we must have

$$\mathbb{1} = (\mathbb{M} + \delta\mathbb{M})(\mathbb{M}^{-1} + \delta(\mathbb{M}^{-1})) = 1 + \delta\mathbb{M}\mathbb{M}^{-1} + \mathbb{M}\delta(\mathbb{M}^{-1}) + \text{small},$$

which implies

$$\delta\mathbb{M}\mathbb{M}^{-1} = -\mathbb{M}\delta(\mathbb{M}^{-1}).$$

Applying this to the metric, and expressing it in index notation, we obtain

$$\delta g^{\mu\nu} = -g^{\mu\rho}\delta g_{\rho\sigma}g^{\sigma\nu}, \qquad \text{and} \qquad \delta g_{\mu\nu} = -g_{\mu\rho}\delta g^{\rho\sigma}g_{\sigma\nu}.$$

To find the other components we use the claim

$$\ln\det\mathbb{M} = \operatorname{Tr}\ln\mathbb{M},$$

In the above expression, we define the natural logarithm of a matrix by using its Taylor series, so that

$$\ln(\mathbb{1} + \mathbb{a}) = \mathbb{a} - \frac{1}{2}\mathbb{a}^2 + \frac{1}{3}\mathbb{a}^3 - \frac{1}{4}\mathbb{a}^4 + ...$$

Then, using linear algebra rules we can prove this statement to be true. With this in place, we see that

$$\delta\sqrt{\det\mathbb{M}} = \sqrt{\det(\mathbb{M} + \delta\mathbb{M})} - \sqrt{\det\mathbb{M}},$$

then using our trick we can say that

$$\delta\sqrt{\det\mathbb{M}} = e^{\frac{1}{2}\ln\det(\mathbb{M}+\delta\mathbb{M})} - \sqrt{\det\mathbb{M}} = e^{\frac{1}{2}\operatorname{Tr}\ln(\mathbb{M}+\delta\mathbb{M})} - \sqrt{\det\mathbb{M}},$$

then in the following step, we can Taylor expand the natural logarithm to first order in the perturbation, and then similarly Taylor expand the exponential function:

$$\delta\sqrt{\det\mathbb{M}} = e^{\frac{1}{2}\operatorname{Tr}\ln\mathbb{M}}e^{\frac{1}{2}\operatorname{Tr}\mathbb{M}^{-1}\delta\mathbb{M}} - \sqrt{\det\mathbb{M}} = e^{\ln\sqrt{\det\mathbb{M}}}\left(1 + \frac{1}{2}\operatorname{Tr}\left(\mathbb{M}^{-1}\delta\mathbb{M}\right)\right) - \sqrt{\det\mathbb{M}},$$

which gives us

$$\sqrt{\det(\mathbb{M})} = \frac{1}{2}\sqrt{\det\mathbb{M}}\operatorname{Tr}\left(\mathbb{M}^{-1}\delta\mathbb{M}\right). \tag{4.17}$$

Using this we can now come up with similar expressions in index notation, $\delta\sqrt{-\det g}$, so we have

$$\delta\sqrt{-\det g} = \frac{1}{2}\sqrt{-\det g}g^{\mu\rho}\delta g_{\rho\mu} = -\frac{1}{2}g_{\mu\nu}\delta g^{\mu\nu}\sqrt{-\det g}, \tag{4.18}$$

In equation 4.16 we will prove the third term vanishes, involving $\delta R_{\mu\nu}$ to vanish. In order to prove this, we have to define numerous rules. To begin, we see that

$$\partial_\mu\sqrt{-\det g} = \frac{1}{2}\sqrt{-\det g}g^{\rho\sigma}\partial_\mu g_{\rho\sigma}. \tag{4.19}$$

Which comes directly from equation 4.18, simply using the similarity between the ideas of a "variation" and a "derivative". We are able to prove that

$$\partial_\mu \sqrt{-\det g} = \Gamma^\nu_{\mu\nu} \sqrt{-\det g}, \tag{4.20}$$

using equation 4.8. Then it follows from starting with the equation 4.9 on the left-hand side and using the product rule on the right-hand side that

$$\sqrt{-\det g} \nabla_\mu v^\mu = \partial_\mu \left(\sqrt{-\det g}\, v^\mu \right). \tag{4.21}$$

In addition, we can directly use the rules for computing covariant derivatives and the formula for the Christoffel symbol to show that

$$\nabla_\mu g_{\nu\rho} = 0. \tag{4.22}$$

With this in place, since $\nabla_\mu (g_{\nu\rho} g^{\rho\sigma}) = \nabla_\mu (\mathbb{1}) = 0$, we can show that

$$\nabla_\mu g^{\nu\rho} = 0. \tag{4.23}$$

One can also show that

$$\begin{aligned} \delta R_{\rho\nu} &= \partial_\beta \left(\delta \Gamma^\beta_{\rho\nu} \right) - \partial_\nu \left(\delta \Gamma^\beta_{\beta\rho} \right) + \left(\Gamma^\alpha_{\rho_\nu} \delta \Gamma^\beta_{\alpha\beta} + \Gamma^\beta_{\alpha\beta} \delta \Gamma^\alpha_{\rho\nu} \right) - \left(\Gamma^\alpha_{\rho\delta} \delta \Gamma^\beta_{\alpha\nu} + \Gamma^\beta_{\alpha\nu} \delta \Gamma^\alpha_{\rho\beta} \right), \\ &= \underbrace{\left(\Gamma^\alpha_{\rho_\nu} \delta \Gamma^\beta_{\alpha\beta} - \partial_\nu \left(\delta \Gamma^\beta_{\beta\rho} \right) \right)}_{\nabla_\beta \Gamma^\beta_{\rho\nu}} + \underbrace{\left(\partial_\beta \left(\delta \Gamma^\beta_{\rho\nu} \right) + \Gamma^\beta_{\alpha\beta} \delta \Gamma^\alpha_{\rho\nu} - \Gamma^\alpha_{\rho\delta} \delta \Gamma^\beta_{\alpha\nu} - \Gamma^\beta_{\alpha\nu} \delta \Gamma^\alpha_{\rho\beta} \right)}_{-\nabla_\nu \Gamma^\beta_{\beta\rho}}, \end{aligned}$$

giving us

$$\delta R_{\rho\nu} = \nabla_\beta \Gamma^\beta_{\rho\nu} - \nabla_\nu \Gamma^\beta_{\beta\rho}. \tag{4.24}$$

Now we can plug in the result for $\delta R_{\mu\nu}$, in terms of the covariant derivatives,

$$\int \sqrt{-\det g} g^{\mu\nu} \delta R_{\mu\nu} d^{3+1}x = \int \sqrt{-\det g} g^{\mu\nu} \left(\nabla_\beta \Gamma^\beta_{\nu\mu} - \nabla_\mu \Gamma^\beta_{\beta\nu} \right) d^{3+1}x,$$

and use the rule about covariant derivatives and $\sqrt{-\det g}$ to turn them into partial derivatives as shown here

$$\int \left(\partial_\beta \left(\sqrt{-\det g} g^{\mu\nu} \Gamma^\beta_{\mu\nu} \right) - \partial_\mu \left(\sqrt{-\det g} g^{\mu\nu} \Gamma^\beta_{\beta\nu} \right) \right) d^{3+1}x.$$

These will all be surface terms, which won't contribute anything to the equation, this is because we assume the perturbation vanishes at the boundaries of space-time.

Now we have everything set up to go back to the action from equation 4.15, and its small perturbation component from equation 4.16. Using equation 4.19, substituting $g^{\mu\nu} R_{\mu\nu}$ with R, and changing our indices we have

$$\delta S = \int_{\mathbb{R}^{3+1}} \sqrt{-\det g} \left(\delta g^{\mu\nu} \left(-\frac{1}{2} g_{\mu\nu} R + R_{\mu\nu} \right) \right) d^{3+1}x.$$

Now to find the Euler-Lagrange equation, we need this component to be equal to zero. At this point, in order for the equation to be equal to zero, the following statement must be true:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 0. \tag{4.25}$$

This is the Euler-Lagrange equation for general relativity or Einstein's equation in a vacuum. If we put the source term $T_{\mu\nu}$ back in, we would have a non-vanishing right-hand side. We will solve equation 4.25 to find the Schwartzschild metric.

4.4 Solving Einstein's Equation

We now want to look for a solution to Einstein's equation that will correspond to the curvature resulting from a single point mass M at the origin. When beginning this derivation we start with assuming spherical symmetry. Meaning that the gravitational field generated by the mass is the same in all directions from its center. This derivation requires the use of Christoffel symbols which will be explained in the next section.

4.4.1 Assumptions about the Metric

Physically it makes sense to look for a solution that is static, and spherically symmetric. But in addition, since the source exists only at the origin, it should satisfy Einstein's equation in a vacuum everywhere else. This along with the requirements that the solution behaves appropriately when far away from the point mass and in the Newtonian limit, actually determines the solution without having to deal with what $T_{\mu\nu}$ is at the origin, where the point mass is located. Since we need a metric that is spherically symmetric and static, the form of the metric would then necessarily need to be

$$ds^2 = -a(r)dt^2 + b(r)dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right), \tag{4.26}$$

where $a(r)$ and $b(r)$ are functions of the radial coordinate which we need to find.

Now we're able to define the metric tensor $g_{\mu\nu}$ as

$$\boldsymbol{g} = \begin{bmatrix} -a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\sin^2\theta \end{bmatrix},$$

and its inverse $g^{\mu\nu}$ as:

$$\boldsymbol{g}^{-1} = \begin{bmatrix} -\frac{1}{a} & 0 & 0 & 0 \\ 0 & \frac{1}{b} & 0 & 0 \\ 0 & 0 & \frac{1}{r^2} & 0 \\ 0 & 0 & 0 & \frac{1}{r^2\sin^2\theta} \end{bmatrix}.$$

We need to figure out what Einstein's equations imply for a metric in this form.

4.4.2 Deriving Christoffel Symbols

Now we will calculate the Riemann curvature tensor, which is given by equation 4.12, and the Christoffel symbols which are given by equation 4.11. We shall proceed to delineate the specific Christoffel symbols that are non-zero. We know that the metric only depends on r and θ, this means that we can ignore the Christoffel symbols that do not depend on r and θ. Since the metric is diagonal, we can ignore the cases where all of the indices of the Christoffel symbol are all different. One of the indices must be either denoted

by r or θ, while the remaining two indices may be any value, as long as they are identical. We also keep in mind that $\Gamma^{\alpha}_{\mu\nu} = \Gamma^{\alpha}_{\nu\mu}$. Another thing to keep in mind is that $\Gamma^{r}_{r\nu}$ must be Γ^{r}_{rr} because in

$$\Gamma^{r}_{r\nu} = \frac{1}{2} g^{rr} \left(\partial_\nu g_{rr} + \partial_r g_{r\nu} - \partial_r g_{r\nu} \right),$$

we can see that the derivative ∂_ν has to be ∂_r in order to be non-zero, any other index will not work in place of ν. The pattern persists as we examine $\Gamma^{\theta}_{\theta\nu}$, which should indeed be $\Gamma^{\theta}_{\theta r}$. Similarly, when considering $\Gamma^{\phi}_{\phi\nu}$, it should be either $\Gamma^{\phi}_{\phi r}$ or $\Gamma^{\phi}_{\phi\theta}$. This arises from the observation that the term $g_{\phi\phi}$ encompasses both an r-dependent component and a θ-dependent component. Regarding the $\Gamma^{t}_{\mu\nu}$ terms, only Γ^{t}_{tr} is present. This particular term arises from the fact that taking a derivative with respect to time yields zero.

This leaves us with the following non-vanishing Christoffel symbols

$$\begin{aligned}
\Gamma^{t}_{tr} &= \frac{1}{2} g^{tt} \partial_r g_{tt} = \frac{a'}{2a}, \\
\Gamma^{r}_{tt} &= \frac{1}{2} g^{rr} \partial_r g_{tt} = \frac{a'}{2b}, \\
\Gamma^{r}_{rr} &= \frac{1}{2} g^{rr} \partial_r g_{rr} = \frac{b'}{2b}, \\
\Gamma^{\theta}_{\theta r} &= \frac{1}{2} g^{\theta\theta} \partial_r g_{\theta\theta} = \frac{1}{r}, \\
\Gamma^{r}_{\theta\theta} &= \frac{1}{2} g^{rr} \partial_r - g_{\theta\theta} = -\frac{r}{b}, \\
\Gamma^{\phi}_{\phi r} &= \frac{1}{2} g^{\phi\phi} \partial_r g_{\phi\phi} = \frac{1}{r}, \\
\Gamma^{r}_{\phi\phi} &= \frac{1}{2} g^{rr} \partial_r - g_{\phi\phi} = -\frac{r \sin^2\theta}{b}, \\
\Gamma^{\phi}_{\phi\theta} &= \frac{1}{2} g^{\phi\phi} \partial_\theta - g_{\phi\phi} = \cot\theta, \\
\Gamma^{\theta}_{\phi\phi} &= \frac{1}{2} g^{\theta\theta} \partial_\theta - g_{\phi\phi} = -\sin\theta\cos\theta,
\end{aligned} \tag{4.27}$$

where the primed terms signify the partial derivatives with respect to r.

4.4.3 Deriving the Ricci Tensor and Scalar

Now we can calculate the Ricci Tensor,

$$R_{\rho\nu} = R^{\beta}_{\rho\beta\nu} = \partial_\beta \Gamma^{\beta}_{\rho\nu} - \partial_\nu \Gamma^{\beta}_{\beta\rho} + \Gamma^{\alpha}_{\rho\nu}\Gamma^{\beta}_{\beta\alpha} - \Gamma^{\beta}_{\nu\alpha}\Gamma^{\alpha}_{\beta\rho},$$

which is a diagonal matrix. In order to prove this we need to show that the non-diagonal elements are zero. Since the Ricci Tensor is symmetric, this means we only need to find half of the elements. We see that

$$R_{tr} = R_{rt} = \partial_\beta \Gamma^{\beta}_{rt} - \partial_t \Gamma^{\beta}_{\beta r} + \Gamma^{\alpha}_{rt}\Gamma^{\beta}_{\alpha\beta} - \Gamma^{\beta}_{t\alpha}\Gamma^{\alpha}_{\beta r},$$

using equation 4.27, we see that the first term must be zero because β must be t, and the second term also must be zero because ∂_t of any of our non-vanishing Christoffel symbols will be zero. For the third term α

must be t, but then β can't be any index that results in a non-zero Christoffel symbol. For the last term, we have to have $\alpha = r,\ \beta = t$ or $\alpha = t,\ \beta = r$, this means the last term becomes $-\Gamma^r_{rt}\Gamma^r_{tr} - \Gamma^t_{rr}\Gamma^r_{tt}$, which is zero. The story is similar for the other five Christoffel symbols,

$$R_{r\theta} = R_{\theta t} = 0, \qquad R_{t\phi} = R_{\phi t} = 0, \qquad R_{r\theta} = R_{\theta r} = 0, \qquad R_{r\phi} = R_{\phi r} = 0, \qquad R_{\phi\theta} = R_{\theta\phi} = 0.$$

Now we can find the diagonal elements of the Ricci Tensor, which can be calculated by plugging in the Christoffel values and paying attention to which Christoffel values could not exist. For R_{tt} we can get that

$$R_{tt} = \frac{a''}{2b} - \frac{a'b'}{4b^2} - \frac{a'^2}{4ab} + \frac{a'}{br}. \tag{4.28}$$

For R_{rr} we have

$$R_{rr} = -\frac{a''}{2a} + \frac{a'^2}{4a^2} + \frac{a'b'}{4ab} + \frac{b'}{br}. \tag{4.29}$$

For $R_{\theta\theta}$ we have

$$R_{\theta\theta} = -\frac{1}{b} + \frac{rb'}{2b^2} - \frac{a'r}{2ab} + 1. \tag{4.30}$$

For $R_{\phi\phi}$ we have

$$R_{\phi\phi} = \sin^2\theta R_{\theta\theta}. \tag{4.31}$$

Now we are able to use the Ricci scalar, given by equation 4.14, to calculate

$$R = R^{\mu\nu}g_{\mu\nu} = g^{tt}R_{tt} + g^{rr}R_{rr} + g^{\theta\theta}R_{\theta\theta} + g^{\phi\phi}R_{\phi\phi},$$

which leaves us with

$$R = -\frac{a''}{ab} + \frac{a'b'}{2ab^2} + \frac{a'^2}{2a^2b} - \frac{2a'}{abr} + \frac{2b'}{b^2r} + \frac{2}{r^2} - \frac{2}{br^2}. \tag{4.32}$$

4.4.4 Applying Einstein's Equations

With the foundation firmly established, we can use Einstein's equation to derive an expression for both a and b. From equation 4.25 we will have

$$R_{tt} - \frac{1}{2}Rg_{tt} = R_{rr} - \frac{1}{2}Rg_{rr} = R_{\theta\theta} - \frac{1}{2}Rg_{\theta\theta} = R_{\phi\phi} - \frac{1}{2}Rg_{\phi\phi} = 0,$$

except at the origin.

In order to derive the Schwartzschild metric, we will use the expressions that contain R_{tt} and R_{rr}, and then verify the expression that contains $R_{\theta\theta}$ is correct by plugging in the functions a and b. If we confirm this, the expression for $R_{\phi\phi}$ will also be correct since it only differs by the $\sin^2\theta$ term.

For the R_{tt} term we can plug in the values from equation 4.28 and 4.32 along with the value for g_{tt}. After simplifying we are left with

$$\frac{b'}{b} + \frac{b}{r} - \frac{1}{r} = 0, \tag{4.33}$$

for the R_{rr} term we can plug in the values from equation 4.29 and 4.32 along with the value for g_{rr}. After simplification we have

$$\frac{a'}{ab} + \frac{1}{rb} - \frac{1}{r} = 0. \tag{4.34}$$

From equation 4.33 we can solve this equation as an ordinary differential equation. Which gives us

$$b = \frac{1}{1 - \frac{\kappa_1}{r}}. \tag{4.35}$$

Here, κ_1 is a constant of integration. The value of b in place, we can insert it into equation 4.34, and again integrate, giving us

$$a = \kappa_2 \left(1 - \frac{\kappa_1}{r}\right). \tag{4.36}$$

Similarly, κ_2 is a different constant of integration. This leaves us with

$$ds^2 = -\kappa_2 \left(1 - \frac{\kappa_1}{r}\right) dt^2 + \left(\frac{1}{1 - \frac{\kappa_1}{r}}\right) dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2. \tag{4.37}$$

We will first verify that the expression that contains $R_{\theta\theta}$ is correct. We will plug in the values from equation 4.30, equation 4.32 which after simplifying will give us

$$-\frac{b'}{b} + \frac{a'}{a} + \frac{ra''}{a} - \frac{a'b'r}{2ab} - \frac{ra'^2}{2a^2} = 0. \tag{4.38}$$

Now we will plug in our formulas for a and b as well as a', a'', and b' which are defined by

$$a' = \frac{\kappa_2\kappa_1}{r^2}, \qquad a'' = -\frac{2\kappa_2\kappa_1}{r^3}, \qquad b' = -\frac{\kappa_1}{\left(1 - \frac{\kappa_1}{r}\right)^2 r^2}.$$

After plugging in these values, we see that the left side of this equation becomes zero, showing that both the expressions that contain $R_{\theta\theta}$ and $R_{\phi\phi}$ are true.

In order to determine the value of the constants κ_1 and κ_2 we notice that, physically, space should be flat when we are very far away from the point mass, this means we must get back $ds^2 = -c^2dt^2 + dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)$, which means that $\kappa_2 = c^2$.

We also notice that if the radius is slightly smaller, we must be able to reproduce Newtonian mechanics. This means that $\kappa_1 = \frac{2GM}{c^2}$. This fact is not obvious up to this point, in fact, it only becomes obvious in the course of the next chapter, where with this choice of κ_1, our orbit equations become approximately the same as the Newtonian orbits in the appropriate limit. Then equation 7.1 becomes

$$ds^2 = -c^2 \left(1 - \frac{2GM}{c^2r}\right) dt^2 + \left(1 - \frac{2GM}{c^2r}\right)^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2, \tag{4.39}$$

With the Schwartzschild metric in place, we can now extremize it. For us, this means exploring the physical properties of the space-time geometry associated with a non-rotating, spherically symmetric mass. This analysis will help us understand black holes and the effect that they have on objects captured in their extreme gravitational fields.

Part II

Application

After setting up our background theory, we will now be able to use it for the application of showing the different orbits that would appear when we have a point-like mass rotating around a massive object. We will separate this problem into three different chapters, each using a new way to think of gravity. We will be looking at how the problem changes when we use Newtonian mechanics, when we use special relativity, and when we use general relativity. We will then plot the orbits that each problem implies [10].

Chapter 5

The Orbit using Newtonian Mechanics

We will begin with the basic physical set-up. We have a single, massive, fixed gravitational mass M located at the origin. We want to know the orbital path of a smaller mass m that is influenced by it.

5.1 Deriving the Orbital Equation

The following section goes into detail about how to derive the orbital equation using Newtonian mechanics. All derivations were done with OpenStax's *University Physics Volume 1* [11].

5.1.1 Energy and Angular Momentum Conservation

We will begin with two fundamental principles in physics, the conservation of energy, and of angular momentum. The conservation of energy states that the total amount of energy in a closed system remains constant over time. Energy cannot be created or destroyed; it can only be transformed from one form to another or transferred between objects. Energy can come in different forms, the forms we focus on are kinetic energy and potential energy. Our energy is given by:

$$E = K + U(r) = \frac{1}{2}m\boldsymbol{v}^2 - \frac{GmM}{r}. \tag{5.1}$$

In this context, we assign the symbol K to denote the kinetic energy, while $U(r)$ represents the potential energy. Additionally, m denotes the mass of the smaller object, M stands for the mass of the stationary massive object, $\boldsymbol{v}$ corresponds to the relative speed between the two objects, r signifies the distance separating them, and G signifies the gravitational constant.

The conservation of angular momentum is a fundamental concept that says that the rotational motion of an object or a system is conserved. In our situation, angular momentum is defined as the cross-product between the distance between the two objects and their linear momentum.

Our expression for angular momentum is given by:

$$\boldsymbol{L} = \boldsymbol{r} \times \boldsymbol{p}, \tag{5.2}$$

here, we use $\boldsymbol{p}$ to signify the momentum in the system.

The conservation of angular momentum implies that the orbit must stay in a single plane (the plane that is perpendicular to $\boldsymbol{L}$). This allows us to define the orbital plane as the xy-plane, and then we choose to use polar coordinates $(r(t), \theta(t))$. As a result, we will redefine our energy and angular momentum equations. Our equation for energy becomes

$$E = \frac{1}{2}m\left(\frac{dr}{dt}\right)^2 + \frac{1}{2}m\left(\frac{d\theta}{dt}\right)^2 - \frac{GmM}{r}.$$

Where we introduce $\frac{dr}{dt}$ as the derivative of the radius with respect to time and $\frac{d\theta}{dt}$ as the derivative of the angle with respect to time. Our expression for angular momentum can then also be given by

$$\boldsymbol{L} = mr^2\frac{d\theta}{dt}\hat{\boldsymbol{k}} \equiv L\hat{\boldsymbol{k}}.$$

5.1.2 Non-dimensionalizing the Problem

We can now nondimensionalize the problem, based on the units of the variables we are using. We can define the natural length scale, natural time scale, natural energy scale, and natural angular momentum scale as

$$r_0 = \frac{GM}{c^2}, \qquad t_0 = \frac{r_0}{c} = \frac{GM}{c^3}, \qquad E_0 = mc^2, \qquad mr_0c = \frac{GMm}{c},$$

respectively. Resulting in the following nondimensionalized variables for the radius, time, energy, and angular momentum respectively:

$$\rho = \frac{r}{r_0}, \qquad T = \frac{t}{t_0}, \qquad \mathcal{E} = \frac{E}{mc^2}, \qquad \ell = \frac{Lc}{GMm}. \tag{5.3}$$

In this problem there is no natural speed scale, meaning no obvious choice for the value of "c." Instead, we will use the natural speed scale that will exist once we take into account special relativity. Where "c" is the speed of light. We also employed the value of "c" to ensure that our nondimensionalized variables align with the variables in forthcoming orbital equations. This means that our calculations will be consistent throughout our work. With these variables in place, we can present the nondimensionalized energy and angular momentum equations. For energy we have

$$\mathcal{E} = \frac{1}{2}\dot{\rho}^2 + \frac{1}{2}\rho^2\dot{\theta}^2 - \frac{1}{\rho}, \tag{5.4}$$

where we used $\dot{\rho}$ and $\dot{\theta}$ to represent $\frac{d\rho}{dT}$ and $\frac{d\theta}{dT}$. For angular momentum, we have

$$\ell = \rho^2\dot{\theta}. \tag{5.5}$$

5.1.3 Solving for the Orbital Shape

If we plug in the value of ℓ into $\mathcal{E}$, we can derive an equation for $\rho(t)$ that doesn't involve $\theta(t)$:

$$\dot{\rho} = \sqrt{2}\sqrt{\mathcal{E} - \frac{\ell^2}{2\rho^2} + \frac{1}{\rho}}. \tag{5.6}$$

Now we are able to find an equation of the radius in terms of the angle ($\rho(\theta)$). This is done with the help of noticing that

$$\frac{d\rho}{d\theta} = \frac{d\rho}{dT}\frac{dT}{d\theta} = \frac{\dot{\rho}}{\dot{\theta}} = \frac{\sqrt{2}\sqrt{\mathcal{E} - \frac{\ell^2}{2\rho^2} + \frac{1}{\rho}}}{\ell/\rho^2},$$

we then use separation of variables to integrate the equation, and we are left with

$$\rho(\theta) = \frac{\ell^2}{1 + \sqrt{2\mathcal{E}\ell^2 + 1}\cos(\theta)}. \tag{5.7}$$

5.1.4 Initial Conditions

Next, we want to set up the equation in terms of initial conditions. These initial conditions will be the beginning radial distance, ρ_0, and beginning speed, v_0. We will also choose to start our orbits at apogee or perigee, where the velocity will be exactly perpendicular to the position vector. The reason for setting up this equation in this manner is to ensure easy comparison with the subsequent orbital equations. By maintaining consistency in the initial values of ρ_0 and v_0, all of our plots will be directly comparable. We can define the value $v_0 = \rho_0\dot{\theta}_0$, which means we now have

$$\ell = \rho_0^2\dot{\theta}_0 = \rho_0 v_0,$$

for the angular momentum and

$$\mathcal{E} = \frac{1}{2}\rho_0^2\dot{\theta}_0^2 - \frac{1}{\rho_0} = \frac{1}{2}v_0^2 - \frac{1}{\rho_0},$$

for energy. This gives us the equation

$$\rho(\theta) = \frac{\rho_0^2 v_0^2}{1 + (\rho_0 v_0^2 - 1)\cos\theta}. \tag{5.8}$$

Equation 5.8 possesses distinct solutions that can be characterized as conic sections. This is because this equation is the equation for conic sections, expressed in polar coordinates. Consequently, when graphing this equation, we anticipate observing various orbital patterns, including circular orbits, elliptical trajectories, parabolic paths, and hyperbolic trajectories.

5.2 Results and Visualization

5.2.1 Orbits

In figure 5.1 we can see the different shapes that result from different initial conditions. Figure 5.1a shows the special case for a perfectly circular orbit. Figure 5.1b shows the case where we are still bounded but we're no longer in the special case of having a circular orbit, leaving us with an elliptical orbit. Figure 5.1c shows the special case where we have a parabolic orbit, this is an orbit where the object is close to being bounded but the smaller mass has its trajectory changed by the massive object. Figure 5.1d shows the case where the orbit takes a hyperbolic shape, where the object's trajectory is changed and once again remains unbounded. we can see that the specific outcomes experienced in this problem depend on the initial conditions, ρ_0 and v_0, which are specified in the plot. We will now determine the specific relationships between v_0 and ρ_0 that lead to circular and parabolic orbits

Circular Orbit

We first focus on obtaining an expression for a circular orbit. We achieve this by taking equation 5.8 and equating the right-hand side to be constant. Subsequently, we solve for v_0, resulting in the equation

$$v_0 = \frac{1}{\sqrt{\rho_0}}. \tag{5.9}$$

Parabolic Orbit

Next, we will derive the function that divides the bounded and unbounded orbits, in this circumstance, this is where the orbit is always parabolic. In order to have a parabolic orbit, we look at the case where the orbit is just barely un-bound, requiring $0 < \rho_0 v_0^2 < 2$. We will have

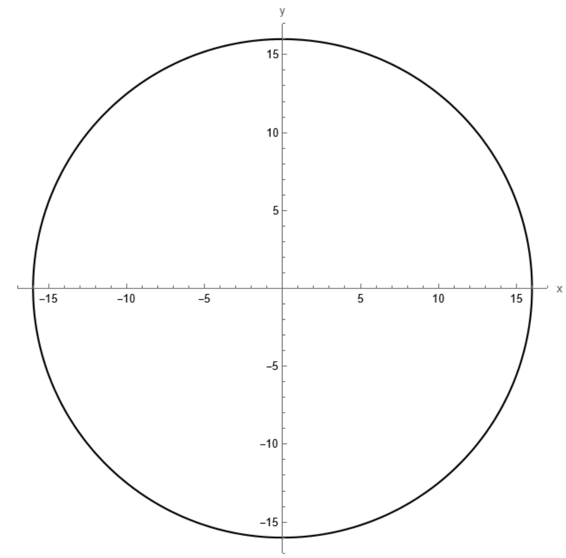
$$v_0 = \sqrt{\frac{2}{\rho_0}}. \tag{5.10}$$

5.2.2 Phase Space Diagram

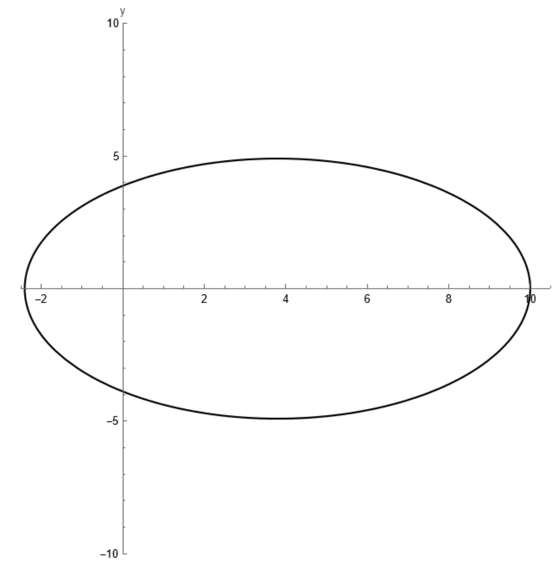
To gain insights into the distinct behaviors of Newtonian gravity, we aim to construct a phase space diagram that visualizes the boundary values between these behaviors. The phase space diagram has the radial component on the x-axis and the velocity component on the y-axis, with the initial conditions on each axis, we will be able to see regions of different orbital behaviors. With our derived functions, we can establish a clear separation between the regions of unbounded and bounded orbits. These functions enable us to plot the corresponding equations and thoroughly examine the areas where bounded, unbounded, and circular orbits manifest themselves. By visualizing these distinct regions, we can gain valuable insights and analyze the behavior of the orbits in a comprehensive manner.

In figure 5.2 we can see at which values of ρ_0 and v_0 the Newtonian orbit causes the object to be bounded and unbounded, as well as the values needed in order to have a circular orbit. This graph contains the equations 5.9 and 5.10, these equations dictate where the boundaries between the bounded and unbounded regions are located based on the initial conditions. This plot has some notable features, it has no infall region, and the reason for this comes from properties of Newtonian mechanics. Due to the conservation of angular momentum, the total angular momentum of the system remains constant over time. As the bodies move along their respective orbits, their individual angular momenta may change, but their sum remains constant. So long as the angular momentum is conserved, the bodies will never collide. For our problem, this means the orbiting object does not fall into the massive object (as long as the orbiting object does not start with zero velocity or starts at the center).

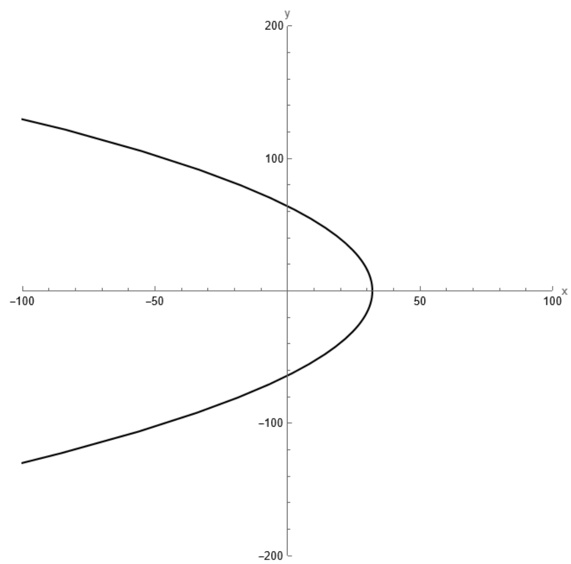
We note that in the Newtonian story, there is no reason to stop the phase space plot at $v_0 = 1$, and Newtonian physics would imply that you *can* make smaller and smaller circular orbits by increasing the speed of the object arbitrarily high. This story will change as we move to special relativity, because any smaller, the velocity would need to be faster than light.



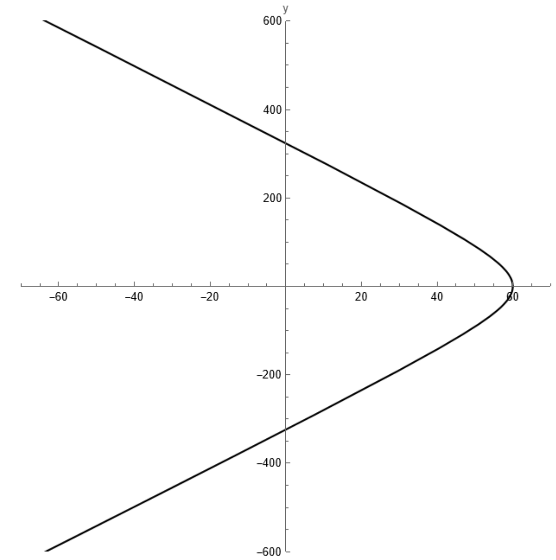
(a) Circular Orbit ($\rho_0 = 16$, $v_0 = 0.25$)



(b) Elliptical Orbit ($\rho_0 = 10$, $v_0 = 0.197$)



(c) Parabolic Orbit ($\rho_0 = 32$, $v_0 = 0.25$)



(d) Hyperbolic Orbit ($\rho_0 = 60$, $v_0 = 0.3$)

Figure 5.1: Four Newtonian orbital plots showing the different orbits that occur based on different initial conditions.

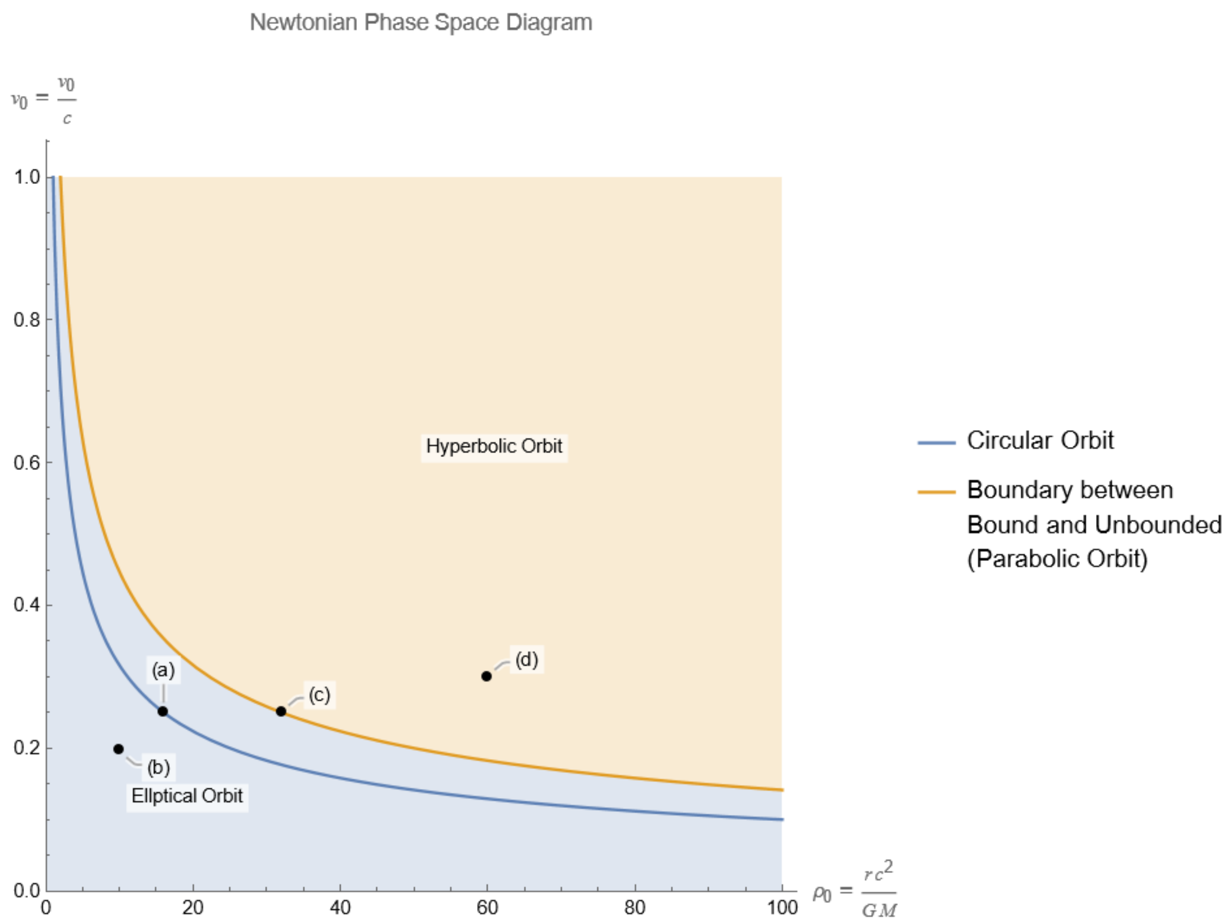


Figure 5.2: Phase space plot showing the regions where an object would be bounded or unbounded for specific values of ρ_0 and v_0. The plot shows the initial conditions of the plots in figure 5.1.

Chapter 6

The Orbit in Special Relativity

In this chapter, we will present the orbits that exist when considering special relativistic mechanics instead of Newtonian mechanics, which was discussed earlier. This change introduces intriguing outcomes and modifications to the problem, particularly regarding the orbits it entails.

6.1 Deriving the Orbital Equation

The following section goes into detail about how to derive the orbital equation using special relativity. Our calculations were performed with the help of Tyler J. Lemmon's, and Antonio R. Mondragon's *Kepler's Orbits and Special Relativity in Introductory Classical Mechanics* [12].

6.1.1 Energy and Angular Momentum Conservation

We will begin in a similar way to the Newtonian case, with energy and angular momentum, this time our kinetic energy term is the relativistic kinetic energy term, $E_k = \gamma mc^2 - m_0c^2$. This term includes the Lorentz factor, γ, which dictates how the kinetic energy will act when approaching relativistic speeds. This shifts the definition of E so that it also includes the mass energy. At low speeds ($v \ll c$) the classical kinetic energy becomes a good approximation for the equation. The relativistic kinetic energy formula reveals that as an object's velocity increases, its kinetic energy not only depends on its mass but also its speed relative to the speed of light. As the speed of the object approaches the speed of light the relativistic kinetic energy increases without bound. Now our energy will be equation 3.18, where as well as the relativistic kinetic energy term we have the potential due to gravity:

$$E = \frac{mc^2}{\sqrt{1 - \frac{v^2}{c^2}}} - \frac{GmM}{r}, \tag{6.1}$$

The relativistic momentum formula similarly includes the Lorentz factor. It goes by $p = \gamma mv$, just as before when considering $v \ll c$, the classical momentum becomes a good approximation. It shows that the momentum of an object not only depends on its mass but also on its speed relative to the speed of light. At relativistic speeds, the momentum increases significantly, and it requires infinite energy to accelerate an object with mass to the speed of light. Our angular momentum term will build off of equation 3.17, where

our classical momentum term is now replaced with the relativistic momentum, giving us

$$\boldsymbol{L} = \boldsymbol{r} \times \boldsymbol{p} = \boldsymbol{r} \times \frac{m\boldsymbol{v}}{\sqrt{1 - \frac{v^2}{c^2}}}. \tag{6.2}$$

Once again we have a situation where because of the conservation of angular momentum, we're able to confine ourselves to the xy-plane. Just as before we can also use polar coordinates. As a result, we are able to redefine our energy, where we split the velocity squared into its components, giving us

$$E = \frac{mc^2}{\sqrt{1 - \frac{v_r^2 + v_\theta^2}{c^2}}} - \frac{GmM}{r}. \tag{6.3}$$

Also, we are able to redefine our angular momentum term, giving us

$$L = \frac{mr^2}{\sqrt{1 - \frac{v_r^2 + v_\theta^2}{c^2}}} \frac{d\theta}{dt}. \tag{6.4}$$

6.1.2 Non-dimensionalizing the Problem

As mentioned before we can use the same nondimensionalized variables in equation 5.3. This will help keep our problems consistent and make it easier to compare each result.

For energy, we will have

$$\mathcal{E} = \frac{1}{\sqrt{1 - \left(\dot{\rho}^2 + \rho^2\dot{\theta}^2\right)}} - \frac{1}{\rho}. \tag{6.5}$$

For angular momentum we have

$$\ell = \frac{\rho^2\dot{\theta}}{\sqrt{1 - \left(\dot{\rho}^2 + \rho^2\dot{\theta}^2\right)}}. \tag{6.6}$$

6.1.3 The Orbital Shape

We will be following the same basic procedure we did for the Newtonian case, where we solved $\dot{\theta}$ in terms of ℓ and ρ, and then substituting that into the other equation, such that we get a single equation for $\rho(t)$ that doesn't involve $\theta(t)$. This results in the equation

$$\dot{\theta} = \frac{\ell}{\rho(\rho\mathcal{E} + 1)}. \tag{6.7}$$

Then in order to find $\dot{\rho}$, the rate the distance changes with time, we need to plug in $\dot{\theta}$ into $\mathcal{E}$, which results in

$$\dot{\rho} = \sqrt{1 - \frac{\rho^2 + \ell^2}{(\rho\mathcal{E} + 1)^2}}. \tag{6.8}$$

Then, just as in the Newtonian case, we find a differential equation for $\rho(\theta)$

$$\frac{d\theta}{d\rho} = \frac{\dot{\theta}}{\dot{\rho}} = \frac{\ell}{\rho\left(1+\mathcal{E}\rho\right)} \Bigg/ \sqrt{1-\frac{\rho^2+\ell^2}{\left(\rho\mathcal{E}+1\right)^2}}.$$

Then we can once again move $d\rho$ to the other side and integrate, and after simplification, we are left with

$$\rho(\theta) = \frac{\frac{\ell^2-1}{\mathcal{E}}}{1+\left(\frac{1}{\mathcal{E}}\right)\sqrt{\left(\mathcal{E}^2-1\right)\left(\ell^2-1\right)+\mathcal{E}^2}\cos\xi\theta}, \tag{6.9}$$

where $\xi = \frac{\sqrt{\ell^2-1}}{\ell}$.

6.1.4 Initial Conditions

We once again will substitute in our initial conditions (ρ_0, v_0), such that we will get new values of the energy and angular momentum, which are not the same as in Newtonian mechanics. Our energy becomes

$$\mathcal{E} = \frac{1}{\sqrt{1-v_0^2}} - \frac{1}{\rho_0},$$

and our angular momentum becomes

$$\ell = \frac{v_0\rho_0}{\sqrt{1-v_0^2}}.$$

We can then substitute these values into $\rho(\theta)$ to get an equation just in terms of the initial conditions. We will then have

$$\rho(\theta) = \frac{\frac{v_0^2\rho_0^2}{1-v_0^2}-1}{\frac{1}{\sqrt{1-v_0^2}}-\frac{1}{\rho_0}+\left(\frac{v_0^2\rho_0}{1-v_0^2}-\frac{1}{\sqrt{1-v_0^2}}\right)\cos\xi\theta}, \tag{6.10}$$

where

$$\xi = \frac{\sqrt{v_0^2\rho_0^2-1+v_0^2}}{v_0\rho_0}. \tag{6.11}$$

There are a couple of important observations to make about this equation. Firstly, the equation maintains the polar form of the conic sections equation, but with a slightly more complicated structure. The most significant difference is the inclusion of the ξ term in the cosine function. This term introduces a slight shift in the cosine value when θ becomes a multiple of 2π. Consequently, when graphed, we expect to observe the precession pattern mentioned in Chapter 1. Furthermore, it's worth noting that if ξ becomes imaginary, which occurs when $v_0^2\rho_0^2 - 1 + v_0^2 < 0$, the cosine term simplifies to:

$$\cos\left(i\tilde{\xi}\theta\right) = \frac{1}{2}\left(e^{\tilde{\xi}\theta}+e^{-\tilde{\xi}\theta}\right).$$

where $\tilde{\xi}$ represents the ξ term under these conditions. Plotting this value reveals an inward spiraling pattern reminiscent of infall, where the smaller object revolving around the larger central body descends into it.

Additionally, we can reproduce the non-relativistic equation from equation 6.10. This occurs when $\xi \approx 1$. Examining equation 6.11, we find that this requires $v_0 \ll 1$ and $v_0\rho_0 \gg 1$. Here, we are arguing that the

relativistic effects kick in when speeds are close to the speed of light, so we want to see what the equation looks like when we consider speeds much less than the speed of light ($v_0 \ll 1$). We also note that if $v_0\rho_0$ is not large, then even if we start with a slow speed compared with the speed of light, at some point in the orbit we will have relativistic speeds. Having $v_0\rho_0 \gg 1$ constrains our problem to be non-relativistic the entire orbit. As a result, the equation simplifies to 5.8.

6.2 Results and Visualization

6.2.1 Orbits

In figure 6.1, we can see the different shapes that result from different initial conditions. Figure 6.1a shows the special case for a perfectly circular orbit. Figure 6.1b shows the case where we are still bounded, but no longer circular, instead of an elliptical orbit we now see precession. We can see that precession essentially is an orbit that shifts before it completes a full orbit. The precession also shows up in figure 6.1c, where the orbit is closer to the circular orbit. We can see that figure 6.1d and figure 6.1e shows the parabolic and hyperbolic trajectories (In reality, these trajectories do not adhere to the mathematical definition of a parabola or a hyperbola. However, for our purposes, we can conceptually treat them as equals). The shapes of these orbits look very similar to the parabolic and hyperbolic orbits in the Newtonian case. We also have figure 6.1f, which shows the new behavior of infall, where the object falls into the center without being able to escape. Once again the two special cases for the boundary between bounded and unbounded, and the circular orbit are something we can solve for. This time, however, the equations will be different, and also we will need to find the boundary between infall and bounded.

Circular Orbit

We will again start with the circular orbit which requires starting with equation 6.10 and setting the right side of the equation as a constant, this means that

$$\rho_0 = \frac{v_0^2\rho_0^2 + v_0^2 - 1}{\sqrt{1 - v_0^2 - \frac{1-v_0^2}{\rho_0}}}.$$

Rearranging this equation, and then using the quadratic formula, we are left with

$$v_0 = \sqrt{\frac{\sqrt{1 + 4\rho_0^2} - 1}{2\rho_0^2}}. \tag{6.12}$$

Parabolic Orbit

We start with equation 6.10, for the boundary between bound and escape. For the orbit to be bound, it requires

$$\sqrt{1 - v_0^2} - \frac{\left(1 - v_0^2\right)}{\rho_0} - \left(v_0^2\rho_0 - \sqrt{1 - v_0^2}\right) > 0,$$

to be satisfied. The boundary between bounded and unbounded will therefore be when the inequality is replaced with an equality. This equation can then be rearranged and simplified. It also requires the use of

the quadratic formula, which upon further simplification gives us

$$v_0 = \sqrt{\frac{1 - 3\rho_0^2 + 2\rho_0^3}{\left(\rho_0^2 - 1\right)^2}}. \tag{6.13}$$

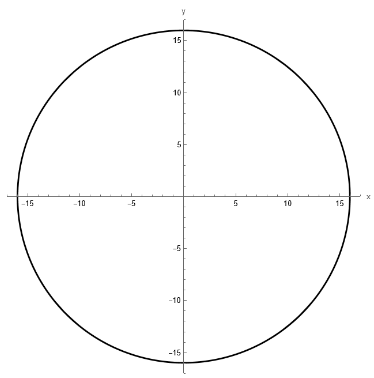
Boundary Between Infall and Bounded

We want to obtain the equation for v_0 that defines the boundary between infall and bound orbits. Looking at equation 6.10, our objective is to find the value of ξ we need to see the transition between a positive value under the square root and a negative one, which leads to an imaginary value, giving us infall. This happens where $\xi = 0$, causing our expression for v_0 to be

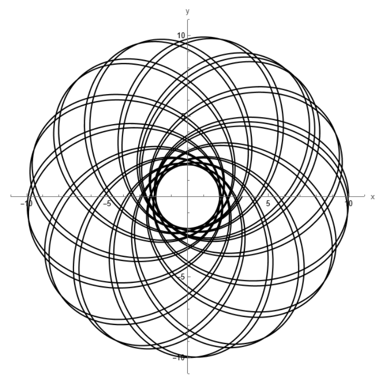
$$v_0 = \sqrt{\frac{1}{\rho_0^2 + 1}} \tag{6.14}$$

6.2.2 Phase Space Diagram

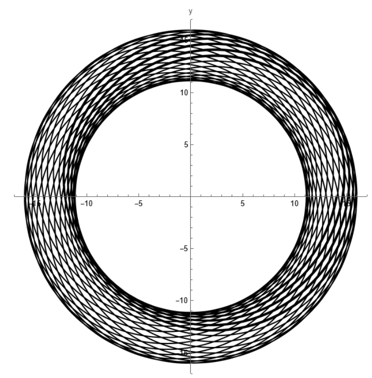
Figure 6.2 contains the equations 6.12, 6.13, and 6.14, these equations dictate where the boundaries between the bounded, unbounded, and infall regions are located based on the initial conditions (ρ_0 and v_0). This phase diagram, unlike the Newtonian one, has to be cut off at $v_0 = 1$, because nothing can achieve a speed greater than that. At a speed arbitrarily close to this, it is possible to find a bound orbit with an arbitrarily small shape. This shows up in the plot since the two boundary lines and the circular orbit line converge at $(0, 1)$. This plot has an infall region, which describes the orbit of the smaller object falling inward to the massive object. This seems to also show that the orbit would be unstable at high speeds and low radii. It is important to note a difference in the physics with the existence of the infall possibility. In our special relativistic orbit, the conservation of angular momentum does not stop a particle from falling into the star, this does not occur in the Newtonian case.



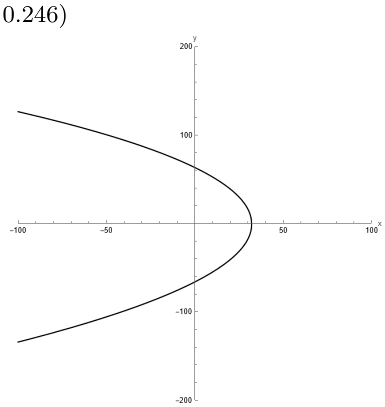
(a) Circular Orbit ($\rho_0 = 16$, $v_0 = 0.246$)



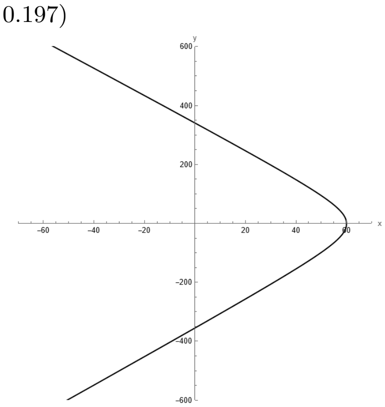
(b) Precessing Orbit ($\rho_0 = 10$, $v_0 = 0.197$)



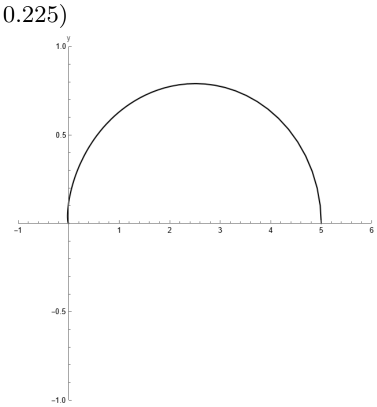
(c) Precessing Orbit ($\rho_0 = 16$, $v_0 = 0.225$)



(d) Parabolic Orbit ($\rho_0 = 32$, $v_0 = 0.244$)



(e) Hyperbolic Orbit ($\rho_0 = 60$, $v_0 = 0.3$)



(f) Infall ($\rho_0 = 5$, $v_0 = 0.1$)

Figure 6.1: Six special relativistic orbital plots showing the different orbits that occur based on different initial conditions.

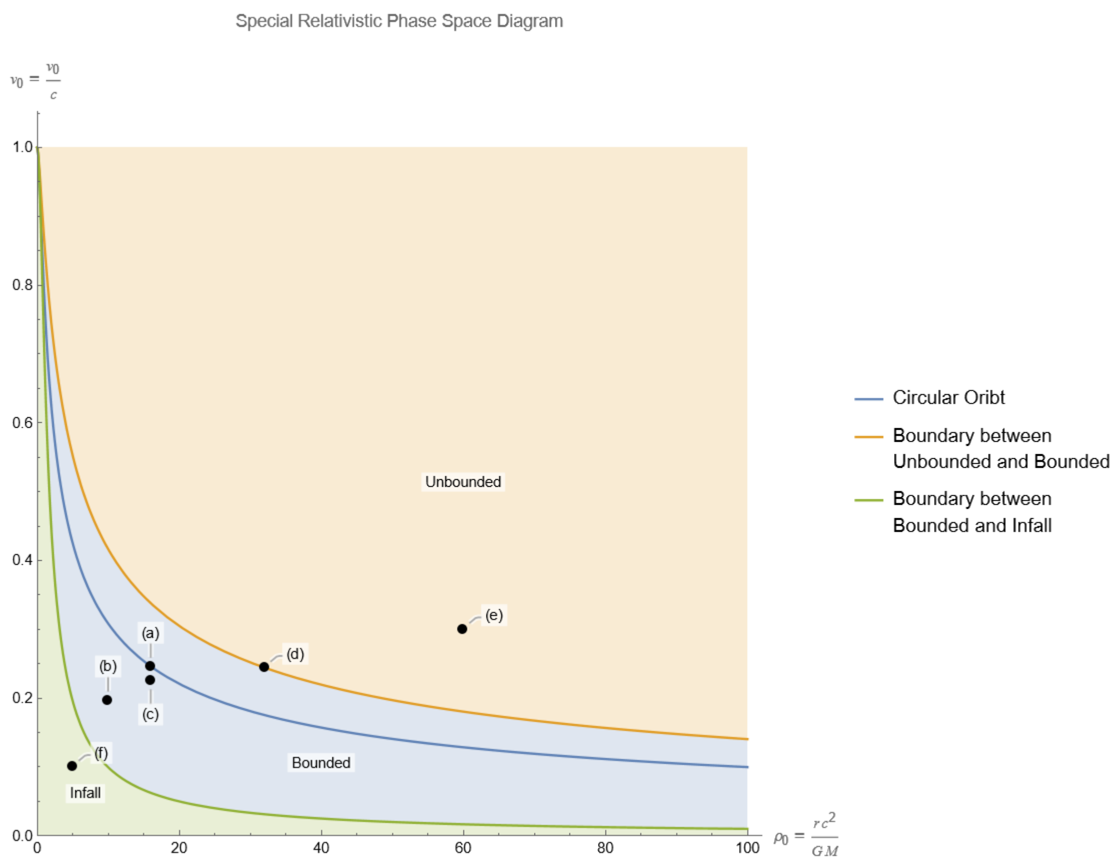


Figure 6.2: Phase space plot showing the regions where an object would experience infall or be bounded or unbounded for specific values of ρ_0 and v_0. The plot shows the initial conditions of the plots in figure 6.1.

Chapter 7

The Orbit in General Relativity

In general relativity, the motion of a particle under the influence of gravity is described by the geodesic equation 4.4, which takes into account the curvature of space-time caused by objects with mass or energy. Here, our problem takes a different form than just a two-body problem. According to general relativity, gravity is not simply a force between objects as described by Newtonian physics. Instead, gravity arises due to the curvature of space-time caused by the presence of mass or energy. This means for our problem, the mass at the origin, M, bends the space-time around it which influences the smaller mass m to orbit around it.

7.1 Deriving the Motion Around a Massive Object

The derivation of the "orbital equation" within the framework of general relativity is a much more complex and intricate process than the other two derivations. It is important to note that there is no analytical expression for the orbital equation in general relativity, rather we will end up with a differential equation which when numerically solved shows the behavior of the orbits. We begin by considering the Schwartzschild metric, a fundamental concept in general relativity. It provides a mathematical description of the curvature of space-time around a spherically symmetric, non-rotating mass. This metric was derived as a solution to Einstein's field equations, as explained in Chapter 4.4.

Our choices for the formulas for energy and angular momentum are not obvious, but they will be conserved. This is because Noether's theorem states that when a system has a continuous symmetry, there is an associated conserved quantity. For our situation, this means time-translation invariance always leads to conservation of energy, and rotational symmetry always leads to conservation of angular momentum. Previously, the formulas changed as we went from non-relativistic to special relativistic energy. As a result, we might expect a change as we go from special relativity to general relativity. Despite the presence of curved space-time, we can still derive equations that describe the energy and the angular momentum. These derived expressions allow us to proceed with calculations similar to those employed in Newtonian and special relativistic contexts. The derivations in this section were performed with the help of Sean Carroll's *Spacetime and Geometry: An Introduction to General Relativity* [4].

7.1.1 The Action and Non-Dimensionalization

We will start with the expression for the Schwartzschild metric, equation 4.39. If we remember our non-dimensionalized variable from equations 5.3, we will have

$$ds^2 = r_0^2 \left(-\left(1-\frac{1}{\rho}\right) dT^2 + \left(1-\frac{1}{\rho}\right)^{-1} d\rho^2 + \rho^2 \left(d\theta^2 \sin^2\theta d\phi^2\right) \right), \tag{7.1}$$

7.1.2 Euler-Lagrange Equations

With the dimensionless form of the Schwartzschild metric in place, we can extremize

$$S = \int d\tau = r_0^2 \int \sqrt{\left(1-\frac{2}{\rho}\right) - \left(1-\frac{2}{\rho}\right)^{-1} \left(\frac{d\rho}{dT}\right)^2 - \rho^2 \left(\left(\frac{d\theta}{dT}\right)^2 + \sin^2\theta \left(\frac{d\phi}{dT}\right)^2 \right)} dT. \tag{7.2}$$

Here, our Lagrangian is

$$\mathcal{L} = \sqrt{\left(1-\frac{2}{\rho}\right) - \left(1-\frac{2}{\rho}\right)^{-1} \dot{\rho}^2 - \rho^2 \left(\dot{\theta}^2 + \sin^2\theta \dot{\phi}^2\right)}.$$

Now we find the expression for the following Euler-Lagrange equations:

$$\begin{aligned} \frac{d}{dT}\frac{\partial \mathcal{L}}{\partial \dot{\phi}} - \frac{\partial \mathcal{L}}{\partial \phi} &= 0, \\ \frac{d}{dT}\frac{\partial \mathcal{L}}{\partial \dot{\theta}} - \frac{\partial \mathcal{L}}{\partial \theta} &= 0, \\ \frac{d}{dT}\frac{\partial \mathcal{L}}{\partial \dot{\rho}} - \frac{\partial \mathcal{L}}{\partial \rho} &= 0. \end{aligned} \tag{7.3}$$

The results are

$$\frac{\rho^2 \sin^2\theta \dot{\phi}}{\mathcal{L}} = C, \tag{7.4}$$

$$\frac{d}{dT}\left(\frac{\rho^2 \dot{\theta}}{\mathcal{L}}\right) = -\frac{\rho^2 \sin\theta\cos\theta \dot{\phi}^2}{\mathcal{L}}, \tag{7.5}$$

$$-\frac{d}{dT}\left(\frac{\dot{\rho}}{\mathcal{L}\left(1-\frac{2}{\rho}\right)}\right) = \frac{1}{\mathcal{L}}\left(\frac{1}{\rho^2} + \frac{\dot{\rho}^2}{\rho^2\left(1-\frac{2}{\rho}\right)^2} - \rho\dot{\theta}^2 - \rho\sin^2\theta\dot{\phi}^2\right), \tag{7.6}$$

respectively.

In the previous orbital equations, we were able to confine the motion to an orbital plane. The Schwartzschild metric should have conserved quantities of angular momentum. This arises from the symmetries of the Schwartzschild metric, specifically, the metric is spherically symmetric. We are able to say that the orbit is confined to the $\theta = \frac{\pi}{2}$ plane. Spherical symmetry implies angular momentum is conserved, and the equations of motion we used were consistent with this fact. Our equations now become

$$\tilde{\mathcal{L}} = \sqrt{\left(1-\frac{2}{\rho}\right) - \left(1-\frac{2}{\rho}\right)^{-1} \dot{\rho}^2 - \rho^2\dot{\phi}^2}, \tag{7.7}$$

$$\frac{d}{dT}\left(\frac{\rho^2\dot{\theta}}{\mathcal{L}}\right) = 0, \tag{7.8}$$

$$-\frac{d}{dT}\left(\frac{\dot{\rho}}{\tilde{\mathcal{L}}\left(1-\frac{2}{\rho}\right)}\right) = \frac{1}{\tilde{\mathcal{L}}}\left(\frac{1}{\rho^2} + \frac{\dot{\rho}^2}{\rho^2\left(1-\frac{2}{\rho}\right)^2} - \rho\dot{\phi}^2\right). \tag{7.9}$$

An orbit with $\theta = \frac{\pi}{2}$ automatically satisfies equation 7.5 and so we can drop it.

7.1.3 Energy and Angular Momentum

In equation 7.5, we can see that it can be rewritten as

$$\ell = \frac{\rho^2\dot{\theta}}{\mathcal{L}}, \tag{7.10}$$

here, it makes mathematical sense that ℓ acts as a constant. Physically it is natural to interpret this constant as angular momentum, given its similarities to the angular momentum expressions in the Newtonian and special relativity cases.

Now we will move on to the energy conservation problem, we will reiterate the fact that just from looking at the Euler-Lagrange equations it is not obvious that such an expression should exist. However, the metric is time-independent and that implies a conserved quantity we can associate with energy to this location.

Similarly to the process we followed for the Newtonian and special relativistic cases, we replace terms involving $\dot{\phi}$ with dependence on the constant ℓ. We begin by deriving a fresh expression for $\dot{\phi}$ by solving equation 7.10 for $\dot{\phi}$ and substituting the value from equation 7.7 into it. This results in the equation

$$\dot{\phi} = \sqrt{\frac{\ell^2\left(1-\frac{2}{\rho}\right) - \ell^2\left(1-\frac{2}{\rho}\right)^{-1}\dot{\rho}^2}{\rho^2\left(\rho^2+\ell^2\right)}}. \tag{7.11}$$

Plugging this into equation 7.7, we can get a new expression for $\tilde{\mathcal{L}}$:

$$\tilde{\mathcal{L}} = \sqrt{\left(\frac{\rho^2}{\rho^2+\ell^2}\right)\left(\left(1-\frac{2}{\rho}\right) - \left(1-\frac{2}{\rho}\right)^{-1}\dot{\rho}^2\right)}. \tag{7.12}$$

Now we will convert equation 7.9 to get $\mathcal{E}$. First we'll state that $\frac{\ell^2\tilde{\mathcal{L}}}{\rho^3} = \frac{\rho\dot{\phi}^2}{\tilde{\mathcal{L}}}$, giving us

$$-\frac{d}{dT}\left(\frac{\dot{\rho}}{\tilde{\mathcal{L}}\left(1-\frac{2}{\rho}\right)}\right) = \frac{1}{\tilde{\mathcal{L}}}\left(\frac{1}{\rho^2} + \frac{\dot{\rho}^2}{\rho^2\left(1-\frac{2}{\rho}\right)^2}\right) - \frac{\ell^2\tilde{\mathcal{L}}}{\rho^3}. \tag{7.13}$$

Now we want to find an expression for $\mathcal{E}$, it can be proven that

$$\frac{\rho^2+\ell^2}{\rho^2}\frac{\partial\tilde{\mathcal{L}}}{\partial\rho} = \frac{1}{\tilde{\mathcal{L}}}\left(\frac{1}{\rho^2} + \frac{\dot{\rho}^2}{\rho^2\left(1-\frac{2}{\rho}\right)^2}\right) + \frac{\ell^2\tilde{\mathcal{L}}}{\rho^3}. \tag{7.14}$$

This can be shown by taking the derivative of equation 7.12 with respect to ρ, the radial coordinate, and using the chain rule and the product rule. Plugging this into equation 7.13, we will have

$$-\frac{d}{dT}\left(\frac{\dot{\rho}}{\tilde{\mathcal{L}}\left(1-\frac{2}{\rho}\right)}\right)=\frac{\rho^2+\ell^2}{\rho^2}\frac{\partial\tilde{\mathcal{L}}}{\partial\rho}-\frac{2\ell^2\tilde{\mathcal{L}}}{\rho^3},$$

then we subtract both sides with

$$-\frac{d}{dT}\left(\frac{\dot{\rho}}{\tilde{\mathcal{L}}\left(1-\frac{2}{\rho}\right)}\right)-\frac{\ddot{\rho}}{\tilde{\mathcal{L}}\left(1-\frac{2}{\rho}\right)}=\frac{\rho^2+\ell^2}{\rho^2}\frac{\partial\tilde{\mathcal{L}}}{\partial\rho}-\frac{2\ell^2\tilde{\mathcal{L}}}{\rho^3}-\frac{\ddot{\rho}}{\tilde{\mathcal{L}}\left(1-\frac{2}{\rho}\right)}. \tag{7.15}$$

Now we consider starting with the expression

$$\dot{\tilde{\mathcal{L}}}=\frac{\partial\tilde{\mathcal{L}}}{\partial\rho}\dot{\rho}+\frac{\partial\tilde{\mathcal{L}}}{\partial\dot{\rho}}\ddot{\rho},$$

which comes from the chain rule. We can use this to be able to show the claim that

$$\frac{1}{\dot{\rho}}\left(\frac{\rho^2+\ell^2}{\rho^2}\right)\dot{\tilde{\mathcal{L}}}=\left(\frac{\rho^2+\ell^2}{\rho^2}\right)\frac{\partial\tilde{\mathcal{L}}}{\partial\rho}-\frac{\ddot{\rho}}{\tilde{\mathcal{L}}\left(1-\frac{2}{\rho}\right)}. \tag{7.16}$$

This requires starting from our chain rule equation and solving for $\frac{\partial\tilde{\mathcal{L}}}{\partial\dot{\rho}}$, and then simplifying. Afterward, we plug equation 7.16 into equation 7.15 and multiply both sides by $\dot{\rho}$,

$$-\dot{\rho}\frac{d}{dT}\left(\frac{\dot{\rho}}{\tilde{\mathcal{L}}\left(1-\frac{2}{\rho}\right)}\right)-\frac{\dot{\rho}\ddot{\rho}}{\tilde{\mathcal{L}}\left(1-\frac{2}{\rho}\right)}=\left(\frac{\rho^2+\ell^2}{\rho^2}\right)\dot{\tilde{\mathcal{L}}}-\frac{2\dot{\rho}\ell^2\tilde{\mathcal{L}}}{\rho^3},$$

then we recognize that the object left of the equal sign is actually $-\frac{d}{dT}\left(\frac{\dot{\rho}^2}{\tilde{\mathcal{L}}\left(1-\frac{2}{\rho}\right)}\right)$, and on the right, is actually $\frac{d}{dT}\left(\tilde{\mathcal{L}}\left(\frac{\rho^2+\ell^2}{\rho^2}\right)\right)$. This now gives us

$$-\frac{d}{dT}\left(\frac{\dot{\rho}^2}{\tilde{\mathcal{L}}\left(1-\frac{2}{\rho}\right)}\right)=\frac{d}{dT}\left(\tilde{\mathcal{L}}\left(\frac{\rho^2+\ell^2}{\rho^2}\right)\right),$$

$$\frac{d}{dT}\left(\frac{\dot{\rho}^2}{\tilde{\mathcal{L}}\left(1-\frac{2}{\rho}\right)}+\tilde{\mathcal{L}}\left(\frac{\rho^2+\ell^2}{\rho^2}\right)\right)=0.$$

Integrating this, we see that the right side of the equation is a constant, physically this equation is expressing the conservation of energy. We will express this constant using $\mathcal{E}$, where

$$\mathcal{E}=\frac{\dot{\rho}^2}{\tilde{\mathcal{L}}\left(1-\frac{2}{\rho}\right)}+\tilde{\mathcal{L}}\left(\frac{\rho^2+\ell^2}{\rho^2}\right). \tag{7.17}$$

Our equations for the angular momentum, energy, and Lagrangian are now

$$\ell = \frac{\rho^2\dot{\phi}}{\tilde{\mathcal{L}}}, \qquad \mathcal{E} = \frac{\dot{\rho}^2}{\tilde{\mathcal{L}}\left(1-\frac{2}{\rho}\right)} + \tilde{\mathcal{L}}\left(\frac{\rho^2+\ell^2}{\rho^2}\right), \qquad \tilde{\mathcal{L}} = \sqrt{\left(\frac{\rho^2}{\rho^2+\ell^2}\right)\left(\left(1-\frac{2}{\rho}\right) - \left(1-\frac{2}{\rho}\right)^{-1}\dot{\rho}^2\right)}, \tag{7.18}$$

respectively.

7.1.4 Finding the Orbital Shape Differential Equation.

We can use these equations in a similar fashion to our previous sections to calculate the orbital equation. This time we are using general relativity to find an equation of motion, but we will not be able to solve the equation of motion to find the orbital shape. Much like the previous situations we need to find $\frac{\partial\rho}{\partial\phi}$ using $\frac{\partial\rho}{\partial T}\frac{\partial T}{\partial\phi}$.

We can now arrive at an expression for $\dot{\rho}^2$, which can be found by rearranging equation 7.12. This gives us

$$\dot{\rho}^2 = -\frac{\tilde{\mathcal{L}}^2\left(\rho^2+\ell^2\right)\left(1-\frac{2}{\rho}\right)}{\rho^2} + \left(1-\frac{2}{\rho}\right)^2.$$

We will now take our expression for $\dot{\rho}^2$ and plug it into our expression for $\mathcal{E}$ from equations 7.18. This results in

$$\ell = \frac{\mathcal{E}\rho^2\dot{\phi}}{1-\frac{2}{\rho}}, \qquad \mathcal{E} = \frac{1-\frac{2}{\rho}}{\tilde{\mathcal{L}}}. \tag{7.19}$$

We can rearrange the expression for the angular momentum to find an expression for $\dot{\phi}$, which will then give us an expression for $\dot{\rho}$:

$$\dot{\phi} = \frac{\left(1-\frac{2}{\rho}\right)}{\mathcal{E}\ell\rho^2}, \qquad \dot{\rho} = \frac{\left(1-\frac{2}{\rho}\right)\ell}{\mathcal{E}\rho^2}\rho', \tag{7.20}$$

where ρ' is $\frac{d\rho}{d\phi}$. Now we take the value of $\tilde{\mathcal{L}}$ from equations 7.18 and plug in the value of $\tilde{\mathcal{L}}$ from equations 7.19 and the value of $\dot{\rho}$ from equations 7.20, and rearrange until we have an expression for $\mathcal{E}$. This results in

$$\mathcal{E}^2 = \frac{\left(1-\frac{2}{\rho}\right)\left(\rho^2+\ell^2\right)}{\rho^2} + \frac{\ell^2}{\rho^4}{\rho'}^2. \tag{7.21}$$

We can replace ρ with σ where $\sigma = \frac{1}{\rho}$, for the purposes of making the math less difficult. Equation 7.21 now becomes

$$\mathcal{E}^2 = \ell^2{\sigma'}^2 + (1-2\sigma)\left(1+\sigma^2\ell^2\right).$$

We could try to calculate the expression for ϕ by integrating

$$\frac{d\phi}{d\sigma} = \pm\sqrt{\frac{\ell^2}{\mathcal{E}^2-(1-2\sigma)(1+\sigma^2\ell^2)}},$$

just as we've done in the previous sections, however, this would not result in an analytic answer. The above equation is a first-order differential equation for $\sigma(\theta)$, however, it cannot be solved for analytically. When

the derivative of the equation is taken, it transforms into a second-order differential equation. This equation also cannot be solved analytically, however, it is in a better form for the use of a numerical method. The simple reason for this is that there would be a square-rooted term involved, which makes the numerical calculation more awkward. This is because as we go around an orbit we would be switching back and forth between needing the positive root and the negative root. We will take the derivative of this expression with respect to ϕ to ignore that difficulty, we then have

$$\sigma'' = \frac{1}{\ell^2} + 3\sigma^2 - \sigma. \tag{7.22}$$

This is the equation we will be working with numerically.

7.1.5 Initial Conditions

It has the initial conditions that $\sigma(0) = \frac{1}{\rho_0}$ and $\sigma'(0) = 0$. We will also express ℓ in terms of ρ_0 and v_0, which gives us

$$\ell^2 = \frac{v_0^2 \rho_0^2}{1 - \frac{2}{\rho_0} - v_0^2},$$

which we can use in equation 7.22 to give us

$$\sigma'' = \frac{1 - \frac{2}{\rho_0} - v_0^2}{v_0^2 \rho_0^2} + 3\sigma^2 - \sigma. \tag{7.23}$$

Although this equation cannot be solved analytically, we can still uncover some intriguing insights. One noteworthy observation is that the presence of the $3\sigma^2$ term is the only factor that renders this ordinary differential equation nonlinear. If σ is small, we are able to eliminate this term, then the equation becomes solvable. This is because the Newtonian limit occurs when you are always far away from the source ($\sigma \ll 1$). This limit also requires the initial conditions $\rho_0 \gg 1$, and $v_0 \ll 1$, or that we are always far from the source and always moving at non-relativistic speeds. Under these conditions equation 7.22 becomes

$$\sigma'' \approx \frac{1}{v_0^2 \rho_0^2} - \sigma,$$

which is satisfied by the Newtonian solution, equation 5.8.

7.2 Results, Numerics, and Visualization

7.2.1 Numerics

The method used to plot the graph will be the fourth-order Runge-Kutta numerical method. The code for this was made using Wolfram Mathematica [10] and is shown in figure 7.1. This is a widely used numerical method for solving ordinary differential equations. It is an iterative algorithm that provides an accurate approximation of the solution at discrete time steps.

```
RKFour[F_, x0_, Δt_, n_, max_] := Module[{xm = x0, xn = x0, i, tn = 0, result = Table[0, n + 1], k1, k2, k3, k4},
  result[[1]] = {tn, xn}; (*Initial condition*)
  For[i = 2, i ≤ n + 1, i++, (*Run the loop n times (since we're starting at 2)*)
   xm = xn; (*xm is basically x_{n-1}*)
   k1 = F[xn, tn] Δt; (*The Euler step*)
   k2 = F[xn + k1/2, tn + Δt/2] Δt; (*The midpoint step*)
   k3 = F[xn + k2/2, tn + Δt/2] Δt; (*The refined midpoint step*)
   k4 = F[xn + k3, tn + Δt] Δt; (*The refined backwards Euler step*)
   xn = xm + 1/6 (k1 + 2 k2 + 2 k3 + k4); (*xn is basically x_n, this is the weighted average*)
   tn = tn + Δt; (*tn increases overtime, and should reach NΔt by the end*)
   If[xn[[1]] > max,
    Break[];
   ];
   result[[i]] = {tn, xn} (*Append the time and position*)
  ];
  If[i < n + 1,
   result = Drop[result, -(n + 2 - i)]
  ];
  Return[result]
 ];
```

Figure 7.1: The code used as the fourth order Runge-Kutta numerical method. F represents the function that needs to be solved (for us this is equation 7.23). The variable $x0$ represents the initial conditions, and Δt represents the step size, which was set as 0.05. The variable n represents the number of steps taken, which was set at 7854. Finally, the variable max acts as a cut-off value so the radius does not get too small, it was set at 10^6.

7.2.2 Solutions

In figure 7.2, we can see the different solutions of our differential equation depending on the varying initial conditions. Figure 7.2a shows the familiar special case for a circular orbit. Figure 7.2b shows an area where the orbit precesses, figure 7.2c also shows the precession closer to the circular orbit. Figure 7.2d and figure 7.2e show an estimate of the parabolic orbit and an example of a hyperbolic orbit respectively. Figure 7.2f also shows the infall behavior. We worked through to find the equation for the special case of the circular orbit, however, the other boundary equations we did not calculate.

7.2.3 Orbits

In order to derive the equation for a circular orbit, we begin by substituting $1/\rho_0$ for any occurrences of σ in equation 7.23. Next, we set σ'' to zero since, in a circular orbit, the radius remains constant (σ'=1). Hence, our task is to solve the following equation:

$$0 = \frac{1 - \frac{2}{\rho_0} - v_0^2}{v_0^2 \rho_0^2} + \frac{3}{\rho_0^2} - \frac{1}{\rho_0}.$$

Doing so returns

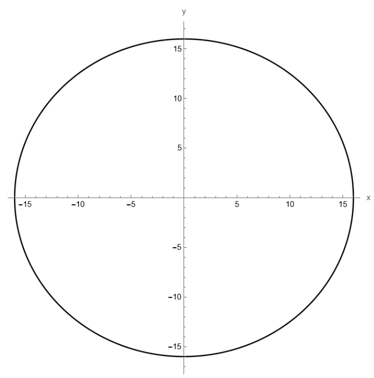
$$v_0 = \frac{1}{\sqrt{\rho_0}}. \tag{7.24}$$

7.3 Phase Space Diagram

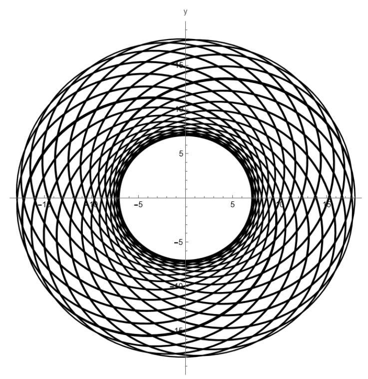
In the case of general relativity, the analytical solution is only feasible for the circular orbital equation. However, for the other regions, the complexity of the equations renders analytical solutions difficult. Consequently, we resort to numerical methods to fill in these regions accurately. To employ the numerical approach, we once again use the Runge-Kutta four numerical method. This time, the process involves randomly selecting a radius and velocity as initial conditions. We then use a variable that takes three different values corresponding to infall, bounded, and unbounded. The variable will take on these three different values depending on if it becomes larger than a maximum value and smaller than a minimum value. We then assign appropriate colors to each point based on its respective region. This process is repeated numerous times, systematically covering the entire phase space diagram. By iteratively applying these steps, we populate the diagram and accurately depict the distinct regions of the orbits. The plot contains 566,658 random initial conditions.

In figure 7.3 we can see at which values of ρ_0 and v_0 the general relativistic orbit causes the object to be bounded, unbounded, or experience infall, as well as the values needed in order to have a circular orbit. This graph contains the equation 7.24, as well as the numerical results for the infall, bounded and unbounded regions. The most striking feature of this plot is that the data are cut off at $\rho_0 < 2$, this is because at the value $\rho = 2$, the coordinates are badly behaved. This is evident from the metric, in equation 7.1. This means we should not trust the results at $\rho_0 = 2$, and we should not start the orbit at $\rho_0 < 2$.

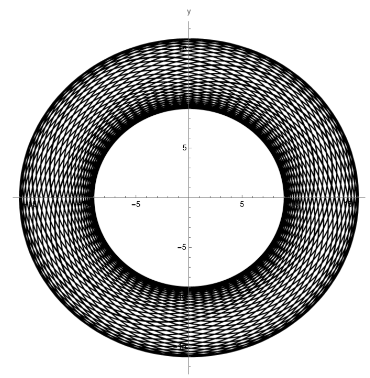
Using equations 5.3 we can trace back the fact that this happens at $r = \frac{2GM}{c^2}$. This value is famously known as the Schwartzschild radius, and below this value, you would be entering the black hole. However, we could create a coordinate system that is smooth at $\rho_0 = 2$, and in fact, locally, space-time is not at all strange at that radius. Then again, once you cross that radius you cannot ever actually escape from the black hole's gravity. We can modify the coordinate system within this region to see the behavior when an object exists within the Schwartzschild radius. However, this was something we didn't explore since our coordinates break down there. This plot also includes a larger infall region compared with figure 6.2, and it has the same circular orbit equation as figure 5.2. At very low radii, we can see that a bounded orbit becomes more unstable, teetering on the edge of infall or unbounded.



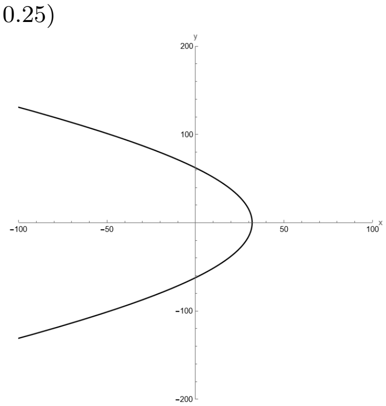
(a) Circular Orbit ($\rho_0 = 16$, $v_0 = 0.25$)



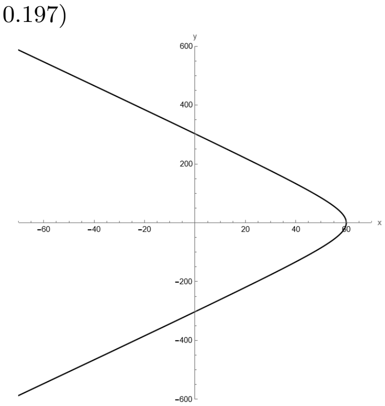
(b) Precessing Orbit ($\rho_0 = 18$, $v_0 = 0.197$)



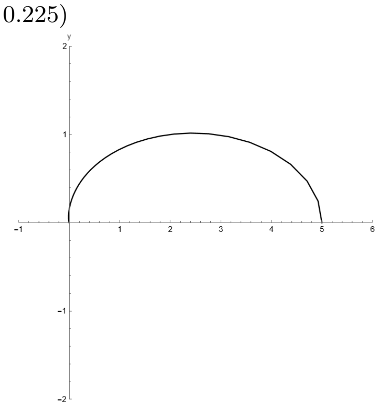
(c) Precessing Orbit ($\rho_0 = 16$, $v_0 = 0.225$)



(d) Parabolic Orbit ($\rho_0 = 32$, $v_0 = 0.25$)



(e) Hyperbolic Orbit ($\rho_0 = 60$, $v_0 = 0.3$)



(f) Infall ($\rho_0 = 5$, $v_0 = 0.1$)

Figure 7.2: Six general relativistic orbital plots showing the different orbits that occur based on different initial conditions. It is important to note that these plots are numerical.

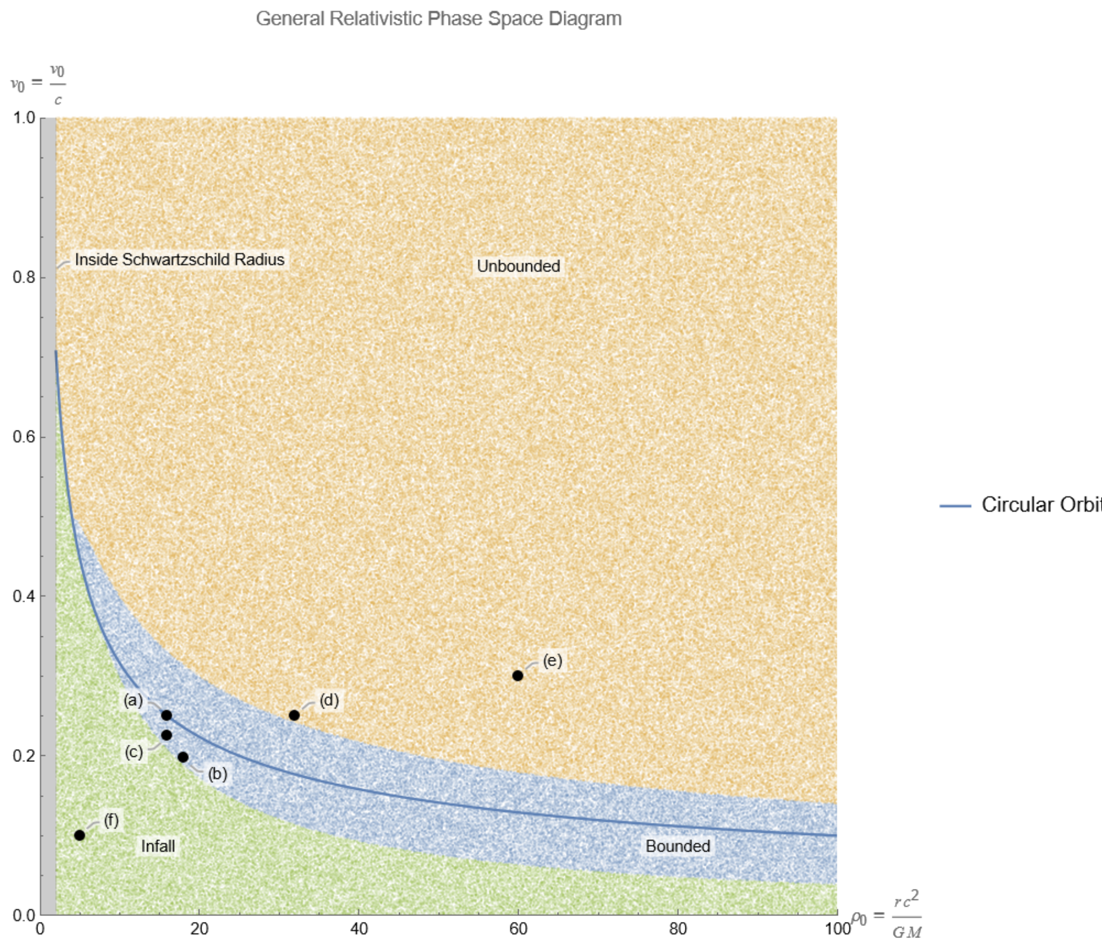


Figure 7.3: Phase space plot showing the regions where an object would experience infall or be bounded or unbounded for specific values of ρ_0 and v_0. This plot also includes a region inside the Schwartzschild radius. The plot shows the initial conditions of the plots in figure 7.2.

Chapter 8

Comparisons

We will now directly compare our results, meaning we will specifically look at the different regions that show up in figure 5.2, figure 6.2, and figure 7.3. We want to delve into a comprehensive comparison of these plots, analyzing the dissimilarities between them.

From figure 8.1 we can see that the infall region for general relativistic orbits is much larger than the region for the special relativistic orbit. This implies that even for low velocities, it is often the case for an object to fall into a more massive object. We also see that special relativity predicts infall in the region $\rho_0 < 2$, which does not exist for general relativity since this is inside of the Schwartzschild radius.

In figure 8.2, we can see that general relativity predicts a bounded region much smaller than both the Newtonian and special relativistic case. Meaning, the general relativistic bounded region is always less than the special relativistic bounded region. These last two details further strengthen the notion that special relativistic orbits may not be entirely trustworthy, as discussed in Chapter 4.1.2. We also see the fact that the Newtonian bounded case exists everywhere below the line from equation 5.10. Once again, general relativity cuts off its data before $\rho_0 = 2$. One interesting thing to notice is that in higher radii and lower speeds the upper bound of the bounded region starts to look similar between all three plots. As we decrease in radii and increase in speed this upper bound starts to diverge. Specifically, we observe that the upper limit of the general relativistic bounded region falls below that of the special relativistic bounded region, while the upper limit of the Newtonian bounded region surpasses both of the aforementioned plots.

From figure 8.3 we can see that the circular orbit for the Newtonian orbit matches the circular orbit of the general relativistic orbit, except for values less than $\rho_0 \leq 2$. And we anticipate this because we know that at lower velocities the Newtonian situation should match with the general relativistic. We also know that general relativity predicts the Schwartzschild radius in the area where data doesn't show up, which explains its cut-off.

We can also see that in larger radii, the circular orbit for special relativity becomes a good approximation, but in lower radii, we can see that the orbit acts differently than how both the Newtonian and the general relativistic scenario describe. The orbit converges closer to zero than expected.

From figure 8.4, we can see that the main differences occur close to the mass and close to the speed of light. It seems like the unbounded region for general relativity was the biggest region out of the three, except for the fact that there is no unbounded region past $\rho_0 = 2$. We can also see that at lower speeds and farther distances the region start becoming good estimates of each other. This gives us confidence in the fact that we expect the general relativistic case to become a good estimate of the Newtonian case when at slow speeds. We also note that the story seems to be true with the special relativistic case as well.

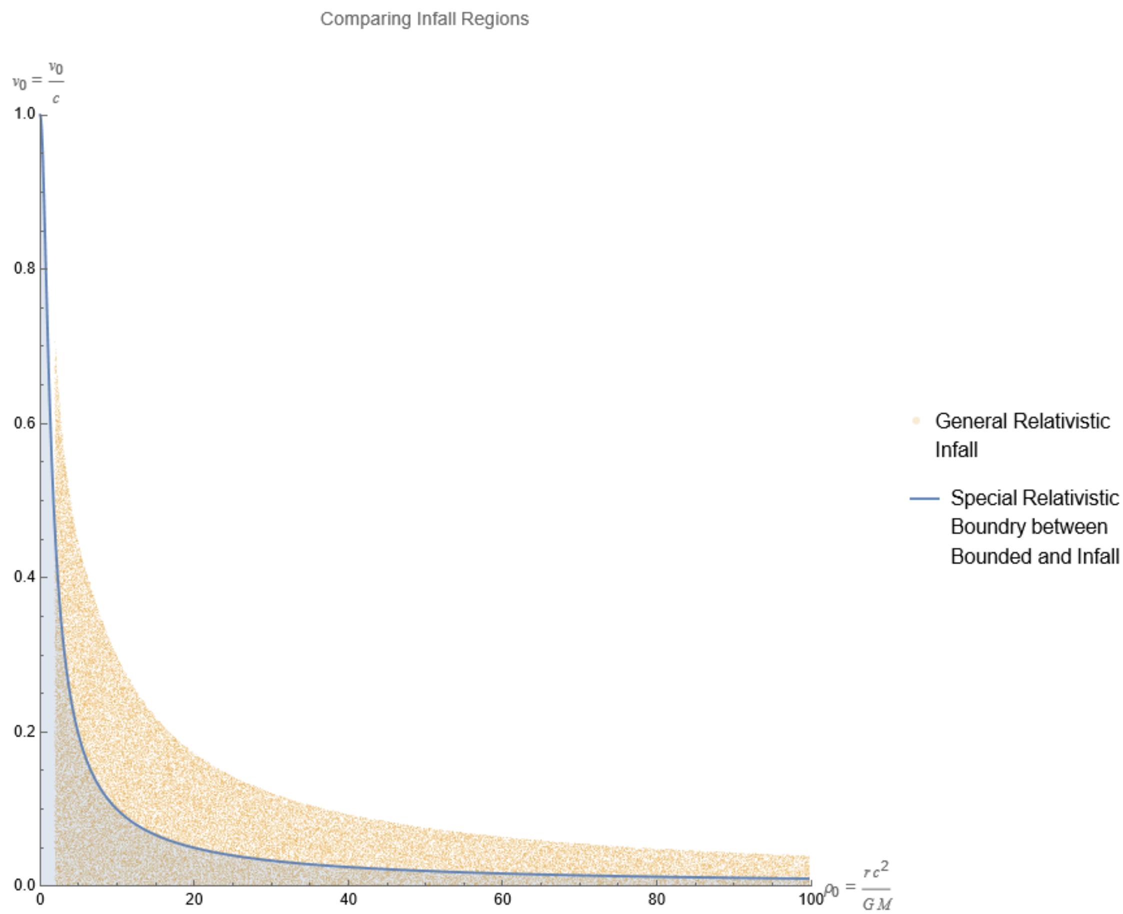


Figure 8.1: Plot comparing the infall regions for both the special and general relativity case.

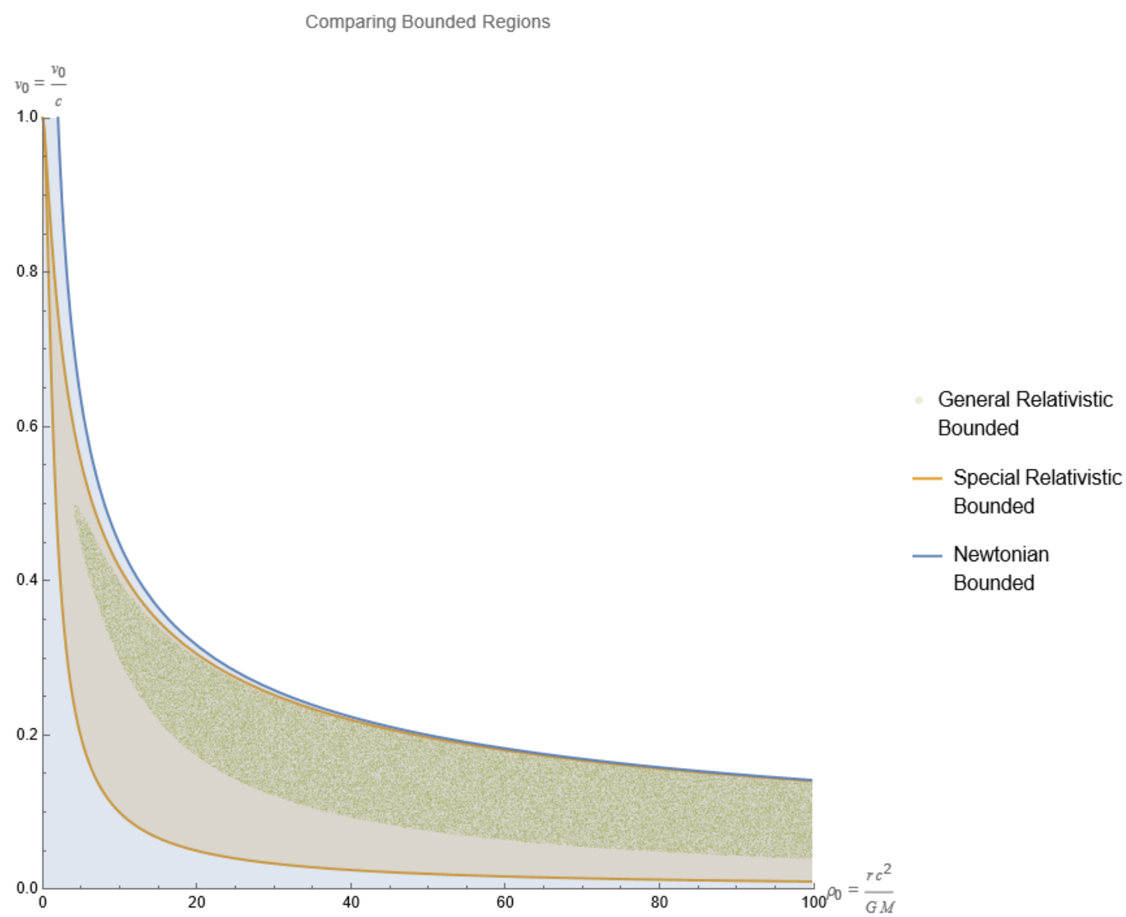


Figure 8.2: Plot comparing the bounded regions for Newtonian, special relativistic, and general relativistic orbits.

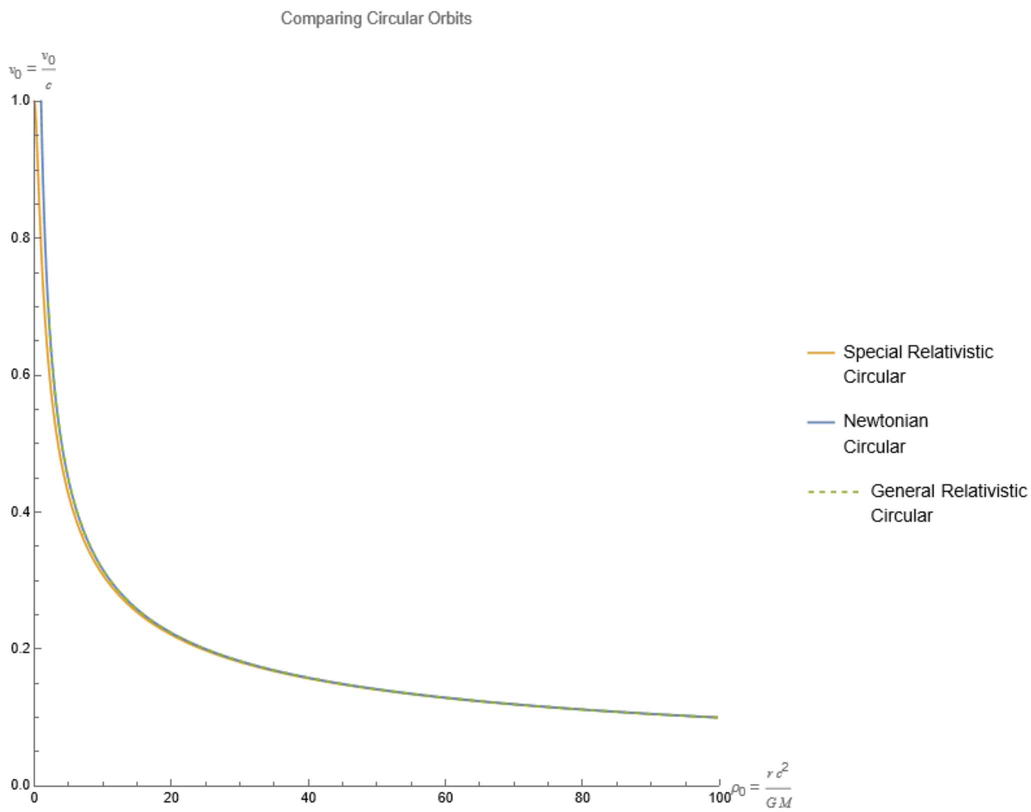


Figure 8.3: Plot comparing the circular orbits for Newtonian, special relativistic, and general relativistic orbits.

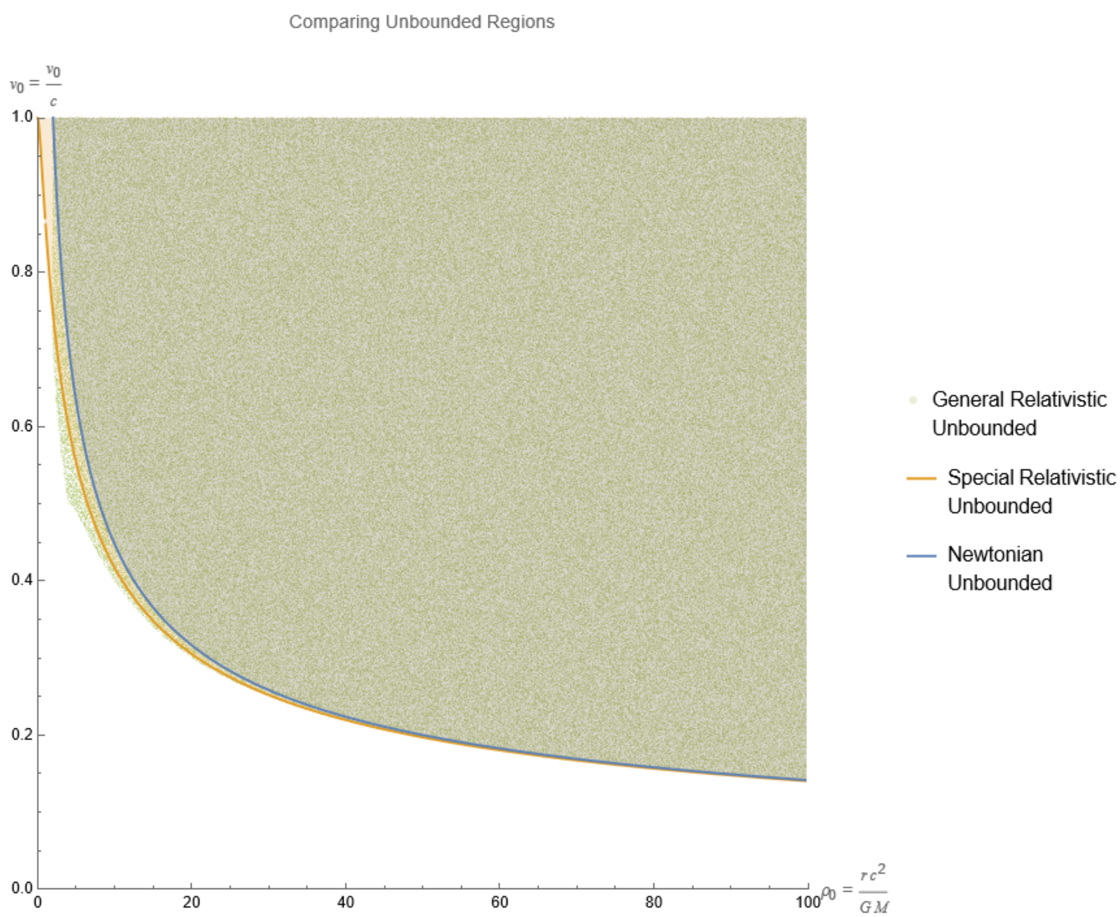


Figure 8.4: Plot comparing the unbounded regions for Newtonian, special relativistic, and general relativistic orbits.

Chapter 9

Conclusion

This thesis has undertaken a comprehensive examination of Newtonian mechanics, special relativity, and general relativity, shedding light on their respective roles and limitations. Through rigorous investigation, we have substantiated the crucial significance of general relativity, particularly in scenarios involving extreme orbits. While acknowledging the practicality of Newtonian mechanics as a reliable approximation for the majority of gravity-related physics encountered in our human experience, we have demonstrated its inadequacy in accurately describing situations characterized by extreme gravitational forces. As we have seen, general relativity implies the existence of infalling orbits, precession, and the Schwartzschild radius. These features do not appear in the Newtonian situation. We have also examined a hypothetical special relativistic orbit, unbounded by the constraints of reality. This unique orbit exhibits notable qualitative resemblances to both the Newtonian and general relativistic cases. However, it also unveils significant divergences, with the absence of a Schwartzschild radius being an important distinction. Additionally, subtle dissimilarities in the phase space diagram further distinguish the special relativistic orbit from its Newtonian and general relativistic counterparts.

For further research, several key avenues warrant exploration. As previously discussed, an area for further investigation involves exploring the region encompassed by the Schwartzschild radius. To achieve this, we would adapt our coordinate system within this region. This can help us gain insights regarding the behavior of objects situated beyond the Schwartzschild radius, an important area of research. We can also analyze more complicated black holes. Ones that include electric charge, angular momentum, or ones that are situated in space that is expanded with respect to time. This would require further calculations but would give us more insight into the behaviors of more realistic black holes. For other areas, we can explore how massless particles (photons) behave in the same situation. This would not be possible using Newtonian physics but could be done with either special or general relativity. Or we could include a line of mass, a plane of mass, and include multiple dimensions. By incorporating these elements into our investigations, we can expand the boundaries of our understanding of gravity.

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