Category Theory and Universal Property

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1 Abstract

Category theory unifies and formalizes the mathematical structure and concepts in a way that various area of interest can be connected. For example, many have learned about the sets and its functions, the vector spaces and its linear transformation, and the group theories and its group homomorphism. Not to mention the similarity of structure in topological spaces, as the continuous function is its mapping. In sum, category theory represents the abstractions of other mathematical concepts. Hence, one could use category theory as a new language to define and simplify the existing mathematical concepts as the universal properties. The goal of this thesis is to provide an understanding of the basic category theory and to derive the universal property of certain mathematical concepts, such as the direct product, the quotient group, and the discrete topology. We start with the basic definitions of Category Theory, namely defining category, functor, natural transformation, and the adjoint. After establishing the basic definition, we will study some notable examples, as well as to propose some interesting examples of our own. Built upon the understanding of the definitions and examples, we will discuss some related questions and come to an application of category theory, the universal property.

2 Preliminary

To discuss the application of category theory, we have to define some of the fundamental concepts (1). Although there exists some differences between various mathematical structures, say set and group, they both have objects and mappings in the language of category theory. After establishing the definition of category, we could study the behavior across fields without referring to its constructive definition.

Definition 1. Category: A Category $\mathscr{A}$ consists of:

1. a collection $ob(\mathscr{A})$ of objects

2. for each $A, B \in ob(\mathscr{A})$, a collection $ob(\mathscr{A}, \mathscr{B})$ of arrows (also called maps and morphism) from A to B

3. for each $A, B, C \in ob(\mathscr{A})$, a function $\mathscr{A}(B,C) \times \mathscr{A}(A,B) \to \mathscr{A}(A,C)$ or $(g, f) \mapsto g \circ f$ called composition

4. for each $A \in ob(\mathscr{A})$, an element 1_A of $\mathscr{A}(A,A)$, called the identity on A

In sum, a category entails objects, arrows (morphismes), composition, and identity that satisfies the associativity and the identity law. Now, we will discuss some of the examples.

1. Category of Set

   (a) Objects are sets. A is a set, and it is the object of **Set**, i.e, $A \in ob(\mathbf{Set})$

   (b) Arrows are ordinary set functions. Given $A, B \in \mathbf{Set}$, then the set function $f : A \to B$ is an arrow of **Set**, i.e, $f \in \mathbf{Set}(A, B)$

   (c) The composition in **Set** is the ordinary composition of set functions, and the idenity function the the identity set function.

   The category set is a fundamental category in the category theory, and we will discuss its connections to many other categories in the later part of this study.

2. Category of Group

   (a) Objects are groups. G is a group, and it is the object of **Grp**, i.e, $G \in ob(\mathbf{Grp})$

   (b) Arrows are group homomorphism. Given $G, H \in \mathbf{Grp}$, then the group homomorphism $h : G \to H$ is the arrow of **Grp**, i.e, $h \in \mathbf{Grp}(G, H)$.

   (c) The composition are the composition of group homomorphism and the identity map is exactly the identity group homomorphism.

   Similarly to the category of set, the category is group is another important category in the thesis. In chapter 3, we will study many examples that have important connection with the category of group.

3. Category of Vector Space

   (a) Objects are vector spaces over a given field. Given a field k, then the vector space V over it is an object of $\mathbf{Vect}_k$, i.e, $V \in ob(\mathbf{Vect}_k)$.

   (b) Arrows are the linear transformations between vector spaces. Given V and W over field k, then the linear map $f : V \to W$ is the arrow $\mathbf{Vect}_k$, i.e, $f \in \mathbf{Vect}_k(V,W)$.

   (c) The compostion are the composition of linear transformations and the identity map is exactly the identity linear map.

   At this point, we should have the intuition of building a category. First, we need objects to construct the category, and then we should be able to come up with the arrows between the objects for the given category. Last but not least, we have to check if the given category satisfies the composition and have an identity map.

4. Category of Topology

   (a) Objects are topological spaces. Given a topological space X, then X is an object of $\mathbf{Top}$, i.e, $X \in ob(\mathbf{Top})$.

   (b) Arrows are the continuous maps between topological spaces. Given $X,Y \in ob(\mathbf{Top})$, then the continuous map $f : X \to Y$ is arrow of $\mathbf{Top}$, i.e, $f \in \mathbf{Top}(X,Y)$.

   (c) The composition are the composition of continuous maps, and the identity is the continuous identity map between topological spaces.

   The category of topology important connections to the other categories. Specifically, one can think of topological space as a set equipped with a certain topological structure. We will explicitly discuss the connection between the category of topology and the category of set.

5. Discrete Category of One Object

(a) There only exists one object in such category, i.e $1 \in ob(\mathbf{1})$.

(b) The arrow is the identity map from the object to itself, i.e. Given 1 in **1**, then the arrow $i : 1 \to 1$ is the arrow of the discrete category, i.e, $i \in \mathbf{1}(1,1)$.

(c) The composition and the identity are vacuously satisfied as there only exists one object and one arrow.

The discrete topology is one of the most trivial examples of all. However, it is a valid category. We will see its appilication later in Chapter 2.

6. Category of Abelian Group

(a) Objects are Abelian groups. A is an Abelian group, and it is the object of **Ab**, i.e, $A \in ob(\mathbf{Ab})$

(b) Arrows are group homomorphism. Given $A, B \in \mathbf{Ab}$, then the group homomorphism $h : A \to B$ is the arrow of **Ab**, i.e, $h \in \mathbf{Ab}(A,B)$

(c) The composition are the composition of group homomorphism and the identity map is exactly the identity group homomorphism.

The Category of Abelian Group **Ab** has similar struture to **Grp**. The only difference is in **Ab**, the object needs to be Abelian. However, for the arrows, composition, and idenity map, **Ab** has almost identical structure to **Grp**.

Definition 2. **Functor**: Let $\mathscr{A}$ and $\mathscr{B}$ be categories. A **functor** $F : \mathscr{A} \to \mathscr{B}$ consists of:

1. a function $ob(\mathscr{A}) \to ob(\mathscr{B})$ or $A \to F(A)$

2. for each $A, A' \in \mathscr{A}$, a function $\mathscr{A}(A,A') \to \mathscr{B}(F(A), F(A'))$ or $f \mapsto F(f)$ satisfying the follwing axioms:

   (a) $F(f' \circ f) = F(f') \circ F(f)$ whenever $A \xrightarrow{f} A' \xrightarrow{f'} A''$ in $\mathscr{A}$

   (b) $F(1_A) = 1_{F(A)}$ whenever $A \in \mathscr{A}$

Functor is built upon the definition of category, as it maps the objects and the arrows between one category to another category. Firstly, let's consider some examples.

1. Free Functor from **Set** to **Top**

   (a) **Set** and **Top** are two categories, and the functor F : **Set** $\to$ **Top** consists of a function $i : S \to D(S)$ that equips set S with the discrete topology, $i(s) = s$, $\forall s \in S$. Recall that the discrete topology defines all subsets to open.

   (b) $f \to F(f)$, define where f is mapping between the sets and $F(f)$ is the continuous map in the topological space

   (c) F satisfies the following axioms:

      i. the composition in the set is preserved by the functor in the topological space

      ii. the identity in the set is preserved by the functor in the topological space

   The functor from **Set** to **Top** is referred as free functors. Informally, free functor is the functor that equip the domain category with minimal mathematical structure to the codomain category. In this case, this specific functor equips the set, the object, with discrete topology, and maps the function of set, the arrow, to the continuous map of topological spaces. We will study another case of free functor.

2. Free Functor from **Set** to **Grp**

   (a) **Set** and**Grp** are two categories, and the functor F : **Set** $\to$ **Grp** equips the set with group structure.

   (b) Given $x \in$ **Set**, F is defined by $\{x\} \mapsto \{x, x^{-1}, id, x^{-1}x^{-1}, xxx...\}$. F also maps the ordinary set function in **Set** to the corresponding group homomorphism in **Grp**.

   (c) The composition and the identity are preserved by this free functor.

Whenever there exists a free functor, its dual, the forgetful functor exists. Consider the following examples:

3. Forgetful Functor from **Top** to **Set**

   (a) **Top** and **Set** are two categories, and the functor $U : \mathbf{Top} \to \mathbf{Set}$ forgets the topological struture of topology and maps the topological space to its underlying set.

   (b) Given $T \in \mathbf{Top}$ and $S \in \mathbf{Set}$, the forgetful functor is defined as such: $U : T \mapsto S$.

   (c) The composition and the identity are preserved by forgetful functor.

   Similarly, there exists a forgetful functor that is dual to $F : \mathbf{Set} \to \mathbf{Grp}$.

4. Forgetful Functor from **Grp** to **Set**

   (a) **Grp** and **Set** are two categories and forgetful functor $U : \mathbf{Grp} \to \mathbf{Set}$ forgets the group strutrute of the given group and leaves its underlying set.

   (b) Given $G \in \mathbf{Grp}$ and $S \in \mathbf{Set}$, then the forgetful functor is defined as such: $U : G \mapsto S$.

   (c) The composition and the identity are preserved by U.

   We've studied two pairs of free functors and forgetful functors, where they are dual to each other respectively. There exists an important application to it, and we will discuss it in the later chapter.

Having established the definition of functor and category, we will study some interesting properties of the functor. First, let's consider a non-example.

Lemma 1. *There does not exist a functor $Z : \boldsymbol{Grp} \to \boldsymbol{Ab}$ with the property that $Z(G)$ is the centre of G for all groups G.*

Before proceeding to the proof of the lemma, let's recall some important definitions relevant to this proof.

Group theory is an important area of interest in the study of Category Theory. Concepts like the category of group, the quotient group and abelianization universal properties will occur many times in our discussion. Before proceeding to the proof of the lemma, let us recall some important definitions in relation to the group theory. The following definition relating to group theory comes from *A First Course in Abstract Algebra* (2) and *Contemporary Abstract Algebra* (3).

Definition 3. Group: Let G be a set together with a binary operation that assigns to each order pair (a,b) of elements of G an element in G denoted ab. We say G is a group under this operation if the following three properties are satisfied:

1. Associatetivity: $\forall a,b,c \in G$, $(ab)c = a(bc)$
2. Identity: $\exists e \in G$ such that $ea = ae = a, \forall a \in G$
3. Inverse: $\forall a \in G$, $\exists b \in G$ such that $ab = ba = e$

Informally speaking, group is a set that equipped with a certain binary operation that is closed under the property of associativity, identity, and the inverse. This notion parallels with example 1, since there also exist a functor from **Set** to **Grp**. At this point, after learning the definition of category, we should be able to view definitions in the language of category theory.

Definition 4. Center: The center, $Z(G)$, of a group G is the subset of elements in G that commute with every element of G. In Symbols, $Z(G) = \{a \in G | ax = xa, \forall x \in G\}$

Definition 5. Definition of Symmetric Group: Let A be the finite set $\{1,2,...,n\}, n \in \mathbb{N}$. The group of all permutations of A is the symmetric group on n letters, and is denoted by S_n

Definition 6. **Group Homomorphism**: A homomorphism ϕ from a group G to a group $\bar{G}$ is a mapping from G into $\bar{G}$ that preserves the group operation; that is, $\phi(ab) = \phi(a)\phi(b)$, $\forall a, b \in G$

In relation to the Category theory, group homomorphism is an arrow in the category of group, and it is a mapping that preserves the group operation. Now we will prove the lemma.

Proof. Consider the category $\mathscr{A}$ consists of three objects S_2, S_3, S_3. $f \in \mathscr{A}(S_2, S_3)$, where f is an inclusion mapping, and $g \in \mathscr{A}(S_3, S_2)$, where $g : S_3 \to S_2$ is a mapping. Consider the following diagram:

- $$\begin{tikzcd}
S_2 \arrow[dr, "f"'] & & S_2 \arrow[rr, dotted, bend left, "Z"] & & Z(S_2) \arrow[dr, "Z(f)"'] & & Z(S_2) \\
& S_3 \arrow[ur, "g"'] & & & & Z(S_3) \arrow[ur, "Z(g)"'] &
\end{tikzcd}$$

The map g sends the even permutations of S_3 to the identity permutation of S_2, and sends the odd permutations S_3 to the other permutation of S_2. Hence, $g \circ f$, defined as $g \circ f : S_2 \to S_3 \to S_2$ is an identity function of S_2. However, $Z(f)$: $Z(S_2) \to Z(S_3)$ is a trivial map, thus $Z(g \circ f)$ is a trivial map as well, which implies that the functor Z does not preserve the identity function, as desired. □

This discussion of **Lemma 1** leads to another important aspect of functor behaviors. After all, functor sends arrows in one category to another category, and itself is a function loosely speaking. Thus, the discussion of surjection and injection arises, and we have a very precious definition for that.

Definition 7. **Faithful**: A functor $F : \mathscr{A} \to \mathscr{B}$ is **faithful** (respectively, **full**) if for each $A, A' \in \mathscr{A}$, the function $\mathscr{A}(A, A') \to \mathscr{B}(F(A), F(A'))$ or $f \mapsto F(f)$ is injective (respectively, surjective)

Now, we will analyze a specific case of faithfulness. Let's propose a general example of a functor $F : \mathscr{A} \to \mathscr{B}$ such that F is faithful but there exist distinct maps f_1 and f_2 in $\mathscr{A}$ with $F(f_1) = F(f_2)$

Example: Consider the following example. Let $\mathscr{A}$ and $\mathscr{B}$ be two categories of groups. $A, B \in ob(\mathscr{A})$ and $C \in ob(\mathscr{B})$. They both are discrete categories, as each object in each category only map to itself, i.e, $f_1 = 1_A$, $f_2 = 1_B$, and $f_3 = 1_C$. $F(f_1) = F(f_2) = f_3$.

Claim: The functor F is faithful. Consider the following diagram:

- $$\begin{tikzcd}
A \arrow[d, "f_1"] \arrow[dr, "F"] & \\
A & C \arrow[d, "f_3"] \\
B \arrow[d, "f_2"] \arrow[ur, "F"'] & C \\
B &
\end{tikzcd}$$

Proof. We proceed by verifying the definition of faithful for each mapping. $\phi(A,B)$ is $\emptyset$, so the mapping $\phi(A,B) \to \phi(F(A),F(B))$ is vacuously injective. $\phi(A,A)$ is 1_A and $\phi(F(A),F(A))$ is $1_{F(A)}$, so $\phi(A,A) \to \phi(F(A),F(A))$ is injective. Similarly, $\phi(B,B) \to \phi(F(B),F(B))$ is injective. Hence, the functor F is faithful, as desired. $\square$

This is a trivial yet interesting example. Since it is easy to construct a faithful functor between the category of monoiods, and it will vacuously satisfy some of the property. However, it is still a valid example.

Having established the definition of functor, one question arises: how could we relate two functors, between the same pair of categories, together? Thus, it leads to the concept of natural transformation.

Definition 8. Natural Transformation: Let $\mathscr{A}$ and $\mathscr{B}$ be categories and let $\mathscr{A} \underset{G}{\overset{F}{\rightrightarrows}} \mathscr{B}$ be functors. A natural transformation $\alpha{:}F \to G$ is a family $(F(A) \xrightarrow{\alpha_A} G(A))_{A \in \mathscr{A}}$, such that the square

$$\begin{tikzcd} F(A) \arrow[r, "F(f)"'] \arrow[d, "\alpha_A"'] & F(A') \arrow[d, "\alpha'_A"] \\ G(A) \arrow[r, "G(f)"'] & G(A') \end{tikzcd}$$

commutes. The maps α_A are called the **components** of α.

The square commutative diagram is often referred as the naturality axiom. Now, let's move to the examples of natural transformations.

1. Power Set Functor and Identity Functor

   (a)

   Definition 9. the power set functor P: **Set**$\rightarrow$**Set** maps each set to its power set and each function $f : X \rightarrow Y$ to the map which sends $U \subseteq X$ to its image $f(U) \subseteq Y$.

   In short, the power set functor convert a given set to its power set.

   (b) Let $\mathscr{A}$ and $\mathscr{B}$ be categories of sets, and let $\mathscr{A} \overset{P}{\underset{id}{\Longrightarrow}} \mathscr{B}$ be functors, where P is the power set functor, and id is the identity functor, and there exists a natural transformation from the power set functor to the identity functor. We will draw the commutative diagram to prove that it indeed is a natural transformation.

   (c) Let $X, Y \in ob(\mathscr{A})$, where $\alpha_X : id(X) \rightarrow P(X)$ and $\alpha_Y : id(Y) \rightarrow P(Y)$ are the inclusion maps from the sets to the power sets. Thus, the following diagram commutes:

$$\begin{tikzcd} id(X) \arrow[r, "id(f)"'] \arrow[d, "\alpha_X"'] & id(Y) \arrow[d, "\alpha_Y"] \\ P(X) \arrow[r, "P(f)"'] & P(Y) \end{tikzcd}$$

Thus, the naturality axiom is satisfied.

There exists some similar structures in Category theory. For instance, category of monoids, identity functor, and identity transformation. In the same spirit, there exist another simple concept called the constant functor. We will give the rigirous defintion.

Definition 10. **Constant Functor**: $F : C \to D$ is a functor that maps each objects of the category C to a fixed object $d \in D$ and each morphism of C to the identity morphism of that fixed object.

Essentially, the constant functor sends objects of the domain to the a fixed objects in the codomain, and all the arrows in the dmoain to the idenity arrow of such fixed objects in the codomain. Hence one question arises: Let F and G be constant functor from a category C to a category D. What's the transforamtion from F to G look like? We will consider some simple exemples from the discrete categories to illustrate the point, and then the case of non-discrete categories will become evident.

Type 1, discrete categories : Let $\mathscr{C}$ and $\mathscr{D}$ be two categories, and only $a, b \in \mathscr{C}$ and $c, d \in \mathscr{D}$. The only morphisms in $\mathscr{C}$ and $\mathscr{D}$ are the identity morphisms. Let $\mathscr{C} \overset{F}{\underset{G}{\Longrightarrow}} \mathscr{D}$ be two constant functors. F maps a, b to c, and $1_a, 1_b$ to 1_c. G maps a, b to d, and $1_a, 1_b$ to 1_d.

1. In this case, there will not be any natural transformations, because there does not exist non trivial arrows between the objects in $\mathscr{D}$

This is the most trivial case of all, as there does not exist any arrows in $\mathscr{D}$. Hence, the naturality axiom diagram will not commute.

Type 2, discrete categories: Let $\mathscr{C}$ and $\mathscr{D}$ be two discrete categories with only one objects, a and b respectively. They only arrow in $\mathscr{C}$ is $f = 1_a$, as the only one in $\mathscr{D}$ is 1_b. Let $\mathscr{C} \overset{F}{\underset{G}{\Longrightarrow}} \mathscr{D}$ be two constant functors. Consider the following diagram:

1.
$$\begin{tikzcd} F(a) \arrow[r, "F(f)"'] \arrow[d, "\alpha_a"'] & F(a) \arrow[d, "\alpha_a"] \\ G(a) \arrow[r, "G(f)"'] & G(a) \end{tikzcd}$$

In this diagram, $F(a) = b$, $F(f) = 1_b$, $G(a) = b$, and $G(f) = 1_b$. Hence, the natural transformation α_a is the identity morphism 1_b. In the discrete type of categories, there either does not exist natural transformation, or the natural transformation is the identity morphism. Since there only exists identity arrow in the codomain, then inevitable the natural transformation becomes the identity morphism. At this point, we should realize the connection between natural transformation and the arrows in the codomain category, and it will become edivident in the non-discrete categories.

Type 3, non-discrete categories: In the most generic case, the natural transformation is the morphism in the co-domain category.

Having elucidated the nature of natural transformation, we will revisit the concept of ismorphism in relation to natural transformation by stating an lemma. We will prove it after establish pertinent defintions.

Lemma 2. *Let $\mathscr{A}$ and $\mathscr{B}$ be two categories. Let $\mathscr{A} \overset{F}{\underset{G}{\Longrightarrow}} \mathscr{B}$ be two functors, and let α be a natural transformation. Then α is a natural isomorphism if and only if α_A:$F(A) \to G(A)$ is an isomorphism for all $A \in \mathscr{A}$.*

Definition 11. Functor Categories: Let $\mathscr{A}$ and $\mathscr{B}$ be two categories, then exists a category where functors from $\mathscr{A}$ to $\mathscr{B}$ are the objects and whose maps are the natural transformation between them. This is called a functor category from $\mathscr{A}$ to $\mathscr{B}$ and written as $[\mathscr{A},\mathscr{B}]$ or $\mathscr{B}^{\mathscr{A}}$.

In the functor category, the functors are the objects, and then the natural transformation between the functors are the arrows. Let F and G be two functors from $\mathscr{A}$ to $\mathscr{B}$, and α

to be the natural transformation between F and G. Consider the following commutative diagram as an illustration:

- $$\begin{array}{c} F : \mathscr{A} \to \mathscr{B} \\ \big\downarrow \alpha \\ G : \mathscr{A} \to \mathscr{B} \end{array}$$

Definition 12. Isomorphism:A map $f : A \to B$ in a category $\mathscr{A}$ is an isomorphism if there exists a map $g : B \to A$ in $\mathscr{A}$ s.t. $gf = 1_A$ and $fg = 1_B$.

Informally speaking, it should be noted that the definition isomorphism is relevant only in the level of arrows, and it shouldn't be confused with isomorphim between functors or categories.

Definition 13. Natural Isomorphism:Let $\mathscr{A}$ and $\mathscr{B}$ be categories. A natural isomorphism between functors from $\mathscr{A}$ to $\mathscr{B}$ is an isomorphism in $[\mathscr{A}, \mathscr{B}]$.

Essentially, the isomorphism in functor category occurs at the level of arrow. In addition, since the natural transformation are the arrows in the functor category, we have to work with natural transformation to prove the lemma.

Proof. ($\Rightarrow$) Assume α is a natural isomorphism, then we have to show $\forall A \in \mathscr{A}$, α_A:$F(A) \to G(A)$ is an isomorphism. Let $A \in \mathscr{A}$ be given. We know there exists an isomorphism in $[\mathscr{A}, \mathscr{B}]$, which implies that if α:$F \to G$, then there exists a map α': $G \to F$ s.t $\alpha'\alpha = 1_F$ and $\alpha\alpha' = 1_G$. Since α_A is an arrow from $F(A)$ to $G(A)$ in $[\mathscr{A}, \mathscr{B}]$, then there exists a map $\alpha'(A) : G(A) \to F(A)$ s.t. $\alpha'(A)\alpha(A) = 1_{F(A)}$ and $\alpha(A)\alpha'(A) = 1_{G(A)}$, so $\alpha(A)$:$F(A) \to G(A)$ is an isomorphism, as desired.

($\Leftarrow$) Assume $\forall A \in \mathscr{A}$, α_A:$F(A) \to G(A)$ is an isomorphism, then we have to show α is a natural isomorphism, that is to show if $\alpha : F \to G$, then $\exists$ a map $\alpha' : G \to F$ s.t. $\alpha'\alpha = 1_F$ and $\alpha\alpha' = 1_G$. Since α_A is an isomorphism, then $\exists$ a map α'_A:$G(A) \to F(A)$ s.t.

$\alpha'(A)\alpha(A) = 1_{F(A)}$ and $\alpha(A)\alpha'(A) = 1_{G(A)}$. Since A is arbitary, then we have $\alpha'\alpha = 1_F$ and $\alpha\alpha' = 1_G$, as desired. $\square$

The logic of this proof is straightforward, but it stresses an important aspect of natural transformation. In plain language, if every individual natural transformation in the codomain as arrows are isomorphism, then the natural transformation as a whole is a natural isomorphism.

To study the property of natural transformation, let's study another trivial example. If there exists a category, then we can obtain its dual category simply by reversing the arrow of the category. Thus, one questions arises, what would what the functor between one category and its dual category look like? Will there be any interesting example of natural transformations? First, we will give rigorous definition of dual category.

Definition 14. Dual Category: Every category $\mathscr{A}$ has a dual category $\mathscr{A}^{op}$, defined by reversing the arrow. Formally, $ob(\mathscr{A}^{op}) = ob(\mathscr{A})$ and $\mathscr{A}^{op}(B,A) = \mathscr{A}(A,B)$ $\forall$ objects $A,B \in \mathscr{A}$. Identity stays the same but composition with argument reversed. If $A \xrightarrow{f} B \xrightarrow{g} C$ are maps in $\mathscr{A}^{op}$, then $A \xleftarrow{f} B \xleftarrow{g} C$ are maps in $\mathscr{A}$.

After established the definition, we will also propose a lemma to answer the question and then prove it.

Lemma 3. *Let $\mathscr{C}$ be a category, and then $\mathscr{C}^{op}$ is its dual category. Let $\mathscr{C} \overset{F^{op}}{\underset{id}{\rightrightarrows}} \mathscr{C}^{op}$ be two functors, and we claim there exists a natural transformation α from F^{op} to id.*

Proof. Let $A,B \in \mathscr{C}$ be arbitrary, and there exists $f \in \mathscr{C}(A,B)$. The functor F^{op} will preserve the same object, but reverse the order of $f \in \mathscr{C}(A,B)$. Consider the following diagram:

1. $$\begin{tikzcd} F^{op}(A) \arrow[r, "F^{op}(f)"'] \arrow[d, "\alpha_A"'] & F^{op}(B) \arrow[d, "\alpha_B"] \\ id(A) \arrow[r, "id(f)"'] & id(B) \end{tikzcd}$$

2. In the diagram, $F^{op}(A) = A$, $F^{op}(B) = B$, and $F^{op}(f) = f$ but with reversed order. In this case, the diagram commutes, and the natural transformations are the identity morphisms in $\mathscr{C}^{op}$.

□

In this particular lemma, we discover the natural transformation between the dual functor and the identity functor. **Lemma 3** is pertinent in the general sense. Now, we will study the natural transformation between abelian functor and the idenity functor under the context of **Grp**. Before proving the lemma, it's important to recall that that $G^{ab} = G/[G.G]$, in which $[G,G] = \{xyx^{-1}y^{-1} \| x,y \in G\}$, is the commutator subgroup.

When natural transformation arises, adjoint arises naturally. Adjoint addresses the relation when two functors are of opposite directions. One would be right adjoint to the other, the other would be left adjoint to the one. We will give the rigirous definition of adjoint.

Definition 15. Adjoint: Let $\mathscr{A} \underset{\text{G}}{\overset{\text{F}}{\rightleftarrows}} \mathscr{B}$ be categories and functors. We say that F is **left adjoint** to G, and G is **right adjoint** to F, and write $F \dashv G$, if $\mathscr{B}(F(A),B) \cong \mathscr{A}(A,G(B))$ naturally in $A \in \mathscr{A}$ and $B \in \mathscr{B}$.

To be adjoint, it entails natural isomorphism between the two functors, and it also has to satisfies the naturality axiom.

1. **Remarks on "Naturally"**: An **adjunction** between F and G is a choice of natural isomorphism. Naturally in $A \in \mathscr{A}$ and $B \in \mathscr{B}$ means that there is a specifed bijection for each $A \in \mathscr{A}$ and $B \in \mathscr{B}$, and that it satisfies a naturality axiom

2. To state it, given objects $A \in \mathscr{A}$ and $B \in \mathscr{B}$, the correspondence between $F(A) \to B$ and $A \to G(B)$ is deonoted by a horizontal bar, in both directions:

(a) $(F(A) \xrightarrow{g} B) \mapsto (A \xrightarrow{\bar{g}} G(B))$

(b) $(F(A) \xrightarrow{\bar{f}} B)\ \ (A \xrightarrow{f} G(B))$

so $\bar{\bar{f}} = f$ and $\bar{\bar{g}} = g$. We call $\bar{f}$ the **transpose** of f, and similarly for g. The relations above are precisely the definition of transpose. In addition, the naturality axiom has two parts, that is $q \,\bar{\circ}\, g = G(q) \circ \bar{g}$, $\forall g, q \in \mathscr{B}$, and $f \,\bar{\circ}\, p = \bar{f} \circ F(p)$, $\forall p, f \in \mathscr{A}$.The composition of transpose has to be satisfied.

Now, we will look at some of the important examples of Adjoints. In particular, by nature, Forgetful functors is left ajoint to the free functors.

1. Forgetful and Free Functors in Algebraic structures

   (a) Consider the free functor F and forgetful U between **Ab** and **Grp**.

$$\begin{tikzcd} \mathbf{Ab} \arrow[d, shift left, "U"] \\ \mathbf{Grp} \arrow[u, shift left, "F"] \arrow[u, phantom, "\dashv"] \end{tikzcd}$$

   In this case, the forgetful functor is left adjoint to the free functor. In previous section of functors, we have also covered the free functor and forgetful functor between vector space and set, and it follows the same rule here.

After establishing definition and concrete examples of adjoint, it is important to note that adjunctions can be interpreted from different perspectives. Say from the view of unit and counit, or from the view of initial and terminal object. Before redefining adjoint, we will intruduce the idea of initial and terminal object first.

Definition 16. Initial/Terminal: Let $\mathscr{A}$ be a category. An object $I \in \mathscr{A}$ is **initial** if for every $A \in \mathscr{A}$, there is exactly one map $I \longrightarrow A$. An object $T \in \mathscr{A}$ is **terminal** if for every $A \in \mathscr{A}$, there is exactly one map $A \longrightarrow T$.

Informally speaking, being an initial object means that there only exist one unique map from such object any other object in the given category. Similarly, being a terminal object means that the map from any object to the terminal object is unique. Consider the empty set in the **Set**, where there only exists one trivial map from the empty set to any given set. Similarly, singleton in **Set** is terminal, as for any given object there exists a unique map from the given object to the singleton set.

Now, we will reintroduce the definition of adjunctions in terms of units and counits. **Definition 15** defines adjunctions in terms of transpose and naturality axiom. However, the unit and counit version offers practical ways for us to study adjunction. Even more so with the initial version, where we have many direct applications.

Definition 17. Unit/Counit: Let $\mathscr{A} \underset{\text{G}}{\overset{\text{F}}{\rightleftarrows}} \mathscr{B}$ be categories and functors. Then, we can define the natural transformations in terms of **unit** and **counit**, where the **unit** is $\eta : 1_{\mathscr{A}} \longrightarrow G \circ F$ and **counit** is $\varepsilon : F \circ G \longrightarrow 1_{\mathscr{B}}$

The idea of unit and counit come handy when verifying if two functors are adjoint to each other.

Lemma 4. *Given an adjunction $F \dashv G$ with unit η and counit ε, the following triangles commute, and they are referred as **triangle identities**.*

$$\begin{tikzcd} F \arrow[r, "F_\eta"] \arrow[dr, "1_F"'] & FGF \arrow[d, "\eta F"] \\ & F \end{tikzcd} \qquad \begin{tikzcd} G \arrow[r, "\eta G"] \arrow[dr, "1_G"'] & GFG \arrow[d, "G_\eta"] \\ & G \end{tikzcd}$$

Remarks: If the diagrams of triangle identities commute, then the we know the given functors, say, F and G are adjoint to each other. It's important to note that F_η is the composition of functor F and the unit map η, and this notation applies to ηF, ηG, and G_η. 1_F and 1_G are the identity map. The diagram is straight forward with respect to the unit and the counit map.

Theorem 1. *Take categories and functor $\mathscr{A} \underset{G}{\overset{F}{\rightleftarrows}} \mathscr{B}$. There is a one-to-one correspondence between*

1. *adjunctions between F and G (with F on the left and G on the right)*

2. *pairs* $(1_{\mathscr{A}} \xrightarrow{\eta} GF, FG \xrightarrow{\varepsilon} 1_{\mathscr{B}})$ *of natural transformations satisfying the triangle identities.*

Theorem 1 have its equivalence via the initial version of adjunction in the later sections. Having introduced the theorem 1, the corollary follows.

Lemma 5. *Take categories and functors* $\mathscr{A} \underset{G}{\overset{F}{\rightleftarrows}} \mathscr{B}$*.Then* $F \dashv G$ *if and only if there exist natural transformations* $1 \xrightarrow{\eta} GF$ *and* $FG \xrightarrow{\varepsilon} 1$ *satisfying the triangle identities.*

Before studying adjunction via initial, we will demonstrate certain property of initial objects in relation to adjoints and prove such property.

Lemma 6. *The left adjoints preserve initial objects: that is, if* $\mathscr{A} \underset{G}{\overset{F}{\rightleftarrows}} \mathscr{B}$*,* $F \dashv G$*, and* I *is an initial object of* $\mathscr{A}$*, then* $F(I)$ *is an initial object of* $\mathscr{B}$*. Dually, show that right adjoints preserve terminal objects.*

Proof. Assume $\mathscr{A} \underset{\text{G}}{\overset{\text{F}}{\rightleftarrows}} \mathscr{B}$, $F \dashv G$, and I is an initial object of $\mathscr{A}$. We want tp show $F(I)$ is an initial object of B, that is to show $\forall B \in \mathscr{B}$, $\exists$ exactly one map from $F(I)$ to B. Find some $B \in \mathscr{B}$ such that $G(B) = A$, then we know there exist exactly one map from $I \to G(B)$. Since we have an adjunction between F and G, then we have: $(F(I) \xrightarrow{g} B) \longmapsto (I \xrightarrow{\bar{g}} G(B))$ $\square$

This lemma follows naturally from **Definition 15** and **Definition 16**. Then, we will illustrate how does the adjunction preserves the order of ordered sets.

Claim: Let $\mathscr{A} \underset{\text{G}}{\overset{\text{F}}{\rightleftarrows}} \mathscr{B}$ be order-preserving maps between ordered sets. $\mathscr{A}$ and $\mathscr{B}$ are the categories of posets, and F and G are two order preserving functors between them. The following conditions are equivalent:

1. for all $a \in A$ and $b \in B$, $f(a) \leq b \longleftarrow a \geq g(b)$
2. $a \geq g(f(a))$ for all $a \in A$ and $f(g(b)) \leq b$ for all $b \in B$

Proof. ($\Rightarrow$) Assume $f(a) \leq b$ iff $a \leq g(b)$ for all $a \in A$ and $b \in B$. Given $a \in A$ and $b \in B$, we want to show $a \leq g(f(a))$ and $f(g(b)) \leq b$. Since $f(a) \leq f(b)$, then we have $a \leq g(f(a))$. Since $g(b) \leq g(b)$, then by 1. we have $f(g(b)) \leq b$, as desired.

($\Leftarrow$) Assume $a \leq g(f(a))$ and $f(g(b)) \leq b$ for all $a \in A$ and $b \in B$, we want to show $f(a) \leq b$ iff $a \leq g(b)$ given $a \in A$ and $b \in B$. Assume $f(a) \leq b$, we want to show $a \leq g(b)$. Since $a \leq g(f(a))$, thenwe know $a \leq g(b)$. Assume $a \leq g(b)$, we want to show $f(a) \leq b$. Since $f(g(b) \leq b$, then we know $f(a) \leq b$, as desired. □

3 Universal Properties

Having shown some interesting properties of adjunction, we will interpret adjunction in terms of initial and terminal object. The notion of initial and terminal object is critical in relation to Universal Property, the goal of this research. Since many Universal Property can be viewed as a comma category with either intial or terminal object. Comma category is in the core of this section, and it's defined as follows:

Definition 18. Given categories $\mathscr{A}, \mathscr{B}, \mathscr{C}$ and functors $P : \mathscr{A} \to \mathscr{C}$ and $Q : \mathscr{B} \to \mathscr{C}$, the **comma category** $(P \downarrow Q)$ is a category.

- $$\begin{tikzcd} & \mathscr{B} \arrow[d, "Q"] \\ \mathscr{A} \arrow[r, "P"'] & \mathscr{C} \end{tikzcd}$$

- objects are triples (A, h, B) with $A \in \mathscr{A}, B \in \mathscr{B}$ and $h : P(A) \to Q(B)$ in $\mathscr{C}$.

- maps $(A, h, B) \to (A', h', B')$ are pairs $(f : A \to A', g : B \to B')$ of maps such that the square commutes.

$$\begin{tikzcd} P(A) \arrow[r, "h"] \arrow[d, "P(f)"'] & Q(B) \arrow[d, "Q(g)"] \\ P(A') \arrow[r, "h'"] & Q(B') \end{tikzcd}$$

Remarks: Essentially, the object of comma categories are arrows in $\mathscr{C}$, so we will denote the object as h instead of (A,h,B). Similarly, the arrows of the comma category is the map between two "maps," namely $h \to h'$, so the whole commutative diagram is the arrow.

In order to discuss and eventually derive universal properties, we will introduce an important Theorem, the equivalent statement of **Theorem 1**, and it partly follows from a lemma.

Lemma 7. *Take an adjunction* $\mathscr{A} \underset{G}{\overset{F}{\rightleftarrows}} \mathscr{B}$ *and an object* $A \in \mathscr{A}$. *Then the unit map* $\eta_A : A \to GF(A)$ *is an initial object of* (A).

Theorem 2. *Take categories and functor* $\mathscr{A} \underset{G}{\overset{F}{\rightleftarrows}} \mathscr{B}$. *There is a one-to-one correspondence between*

1. *adjunctions between F and G (with F on the left and G on the right)*

2. *natural transformation* $\eta : 1_{\mathscr{A}} \to GF$ *such that* $\eta_A : A \to GF(A)$ *is initial in* $(A \Rightarrow G)$ *for every* $A \in \mathscr{A}$

One-to-one correspondence in this context means that if we have an adjunction between F and G, then we will be able to find a unique natural transformation that satisfies certain properties. Vice versa, if we have a natural transformation that satisfies certain properties, then we will be able to find an unique adjunction corresponds to it. Thus, we will name the $1 \to 2$ direction as uniqueness, and name the $2 \to 1$ direction as existence.

To illustrate the significance of **Theorem 2**, we will introduce an universal property of quotient group. Consider the constructive definition and the universal property of quotient group.

3.1 Universal Property of Quotient Group

Definition 19. Let G be a group and let H be a normal subgroup of G. The set $G/H = \{aH : a \in G\}$ is a group under the operation $(aH)(bH) = (ab)H$.

Recall from Abstract Algebra, we have the following proposition:

Proposition. Let G be any group, $N \triangleleft G$, and $g : G \to G/N$ the natural homomorphism defined by $g(x) = xN$. Then let $f : G \to H$ be any homomorphism such that $f(N) = \{id\}$. Then there exists a unique homomorphism $h : G/N \to H$ such that $f = h \circ g$.

- In other words, this diagram commutes

$$\begin{tikzcd}
G \arrow[rr, "f"] \arrow[dr, "g"'] & & H \\
& G/N \arrow[ur, dashed, "\exists! h"'] &
\end{tikzcd}$$

Remarks: It is not our goal to establish equivalence between the two definitions, and we want to derive the universal property using **Theorem 2** to show the existence part for the quotient group. Ultimately, we want to be able illustrate the quotient group without referring to its constructive definitions.

Proof. First, given cateories $Grp, \mathscr{D}$, and 1, a one object category, and functors G and U, we will construct the comma category $(G \Rightarrow U)$. We define $\mathscr{D}$ as such:

- Objects: a pair of two groups (N,G), where N is normal to G, i.e, $N \triangleleft G$.
- Arrows: Let (N,G) and $(H,K) \in \mathscr{D}$ be givem, then $f : G \to K$, where $f(N) = \{id_K\}$.

We define the functor U and G to be such:

- U : $\mathbf{Grp} \to \mathscr{D}$ is a functor.
  - $U : K \mapsto (1,K)$
- $G : 1 \to \mathscr{D}$ is another functor, where 1 is one object category.
  - $G : 1 \mapsto (N,G)$

- It implies the following diagram commutes:

$$\begin{tikzcd} & Grp \arrow[d, "U"] \\ 1 \arrow[r, "G"'] & \mathscr{D} \end{tikzcd}$$

We want to find an unique functor F that is left adjoint to U. Thus, by 2 of **Theorem 2**, we have $\eta : 1_{\mathscr{A}} \to U \circ F$ such that $\eta_A : A \mapsto UF(A)$ is initial in the comma category $(G \Rightarrow U)$. Thus, η_A being initial in the comma category precisely implies the following diagram commutes, given that $f : A \mapsto U(B)$ is another arrow of the comma category:

- $$\begin{tikzcd} A \arrow[rr, "\eta_A"] \arrow[dr, "f"'] & & UF(A) \arrow[dl, dotted, "\exists! h"] \\ & U(B) & \end{tikzcd}$$

Consider the functor $F : \mathscr{D} \to Grp$, where is defined as $F : (N, G) \mapsto G/N$, a pair of groups that $N \triangleleft G$ is mapped to its quotient group G/N. Thus, let $A = (N, G)$, we will have the following commutative diagram, given that $f : A \mapsto U(B)$ is another arrow of the comma category:

- $$\begin{tikzcd} (N,G) \arrow[rr, "\eta_A"] \arrow[dr, "f"'] & & (1, G/N) \arrow[dl, dotted, "\exists! h"] \\ & (1,B) & \end{tikzcd}$$

This commutative diagram is analogous to the universal property of quotient group,

- $$\begin{tikzcd} G \arrow[rr, "f"] \arrow[dr, "g"'] & & H \\ & G/N \arrow[ur, dashed, "\exists! h"'] & \end{tikzcd}$$

which implies that for any other projection map f from G, a group to another group H, it has to factor through the quotient group G/N, as desired. $\square$

Since the universal property addresses mainly the existence and the uniqueness properties of a given mathematical concept. In most of the cases, the idea of initial and terminal objects are very important in relation to derive the universal property.

3.2 Universal Property of Abelianization

Now, we will how to derive the universal property of commutator subgroups, which is analogous to the derivation for quotient group. Consider the following two definitions:

Definition 20. $G^{ab} = G/[G,G]$, where $[G,G] = \{xyx^{-1}y^{-1} : x,y \in G\}$ and $[G,G] \triangleleft G$.

Recall from Abstract Algebra, we have the following proposition:

Proposition. Let G be any group, $G^{ab} = G/[G,G]$, and $g : G \to G^{ab}$ the natural homomorphism defined by $g(x) = x[G,G]$. Then G^{ab} satisfies the following property. Let $f : G \to H$ be any homomorphism from G to an Abelian group H. Then there exist a unique homomorphism $h : G^{ab} \to H$ such that $f = h \circ g$.

- In other words, this diagram commutes

$$\begin{tikzcd} G \arrow[rr, "f"] \arrow[dr, "g"'] & & H \\ & G^{ab} \arrow[ur, dashed, "\exists! h"'] & \end{tikzcd}$$

Remarks: At this point, it's easy to notice that the commutative diagram of commutator subgroups and the diagram of quotient groups are almost identical except the Abelian group takes place of the quotient group. We will prove the universal property using **Theorem 2**.

Proof. First, given categories Ab, the category of Abelian group, $\mathscr{D}$, and 1, a one object category, and functors G and U, we will construct the comma category $(U \Rightarrow G)$. We define define $\mathscr{D}$ as such:

- Objects: a pair of two groups (N,G), where N is normal to G, i.e, $N \triangleleft G$, and G/N is Abelian.

- Arrows: Let (N,G) and $(H,K) \in \mathscr{D}$ be given, then $f : G \to K$, where $f(N) = \{id_K\}$.

We define the functor U and G to be such:

- $G : \mathbf{Ab} \to \mathscr{D}$ is a functor.
  - $G : H \mapsto (1, H)$
- $U : 1 \to \mathscr{D}$ is another functor, where 1 is one object category.
  - $U : 1 \mapsto ([G,G], G)$
- It implies the following diagram commutes:

$$\begin{tikzcd} & Ab \arrow[d, "U"] \\ 1 \arrow[r, "G"'] & \mathscr{D} \end{tikzcd}$$

We want to find an unique functor F that is left adjoint to G. Thus, by 2 of **Theorem 2**, we have $\eta_A : A \mapsto GF(A)$ is initial in the comma category $(U \Rightarrow G)$. Thus, η_A being initial in the comma category precisely implies the following diagram commutes, given that $f : A \mapsto G(B)$ is another arrow of the comma category:

- $$\begin{tikzcd} A \arrow[rr, "\eta_A"] \arrow[dr, "f"'] & & GF(A) \arrow[dl, dotted, "\exists ! h"] \\ & G(B) & \end{tikzcd}$$

Consider the functor $F : \mathscr{D} \to Ab$, where is defined as $F : (N, G) \mapsto G/N$, a pair of groups that $N \triangleleft G$ is mapped to its quotient group G/N. Thus, let $A = (N, G)$, we will have the following commutative diagram, given that $f : A \mapsto G(B)$ is another arrow of the comma category:

- $$\begin{tikzcd} (N,G) \arrow[rr, "\eta_A"] \arrow[dr, "f"'] & & (1, G/N) \arrow[dl, dotted, "\exists ! h"] \\ & (1,B) & \end{tikzcd}$$

This commutative diagram is analogous to the universal property of quotient group,

- $$\begin{tikzcd} G \arrow[rr, "f"] \arrow[dr, "g"'] & & H \\ & G^{ab} \arrow[ur, dashed, "\exists ! h"'] & \end{tikzcd}$$

which implies that for any other projection map f from G, a group to another group H, it has to factor through the Abelian group G^{ab}, as desired. $\square$

3.3 Universal Property of Direct Product

We've discussed universal property through initial object, and now we will look at the universal property of direct product, which is established by the property of terminal object. Consider the two following defintions:

Definition 21. Let G_1, G_2 be a collection of groups. The external direct product of G_1, G_2 written as $G_1 \oplus G_2$ is the set of all 2-tuples for which the 2nd component is an element of G_2 and the operation is componentwise.

Recall from Abstract Algebra, we have the following proposition:

Proposition. Let G_1, G_2 be any groups, $G_1 \oplus G_2$ is their external direct product, and g_1, g_2 is the projection maps (i.e., $g_1(a,b) = a$ and $g_2(a,b) = b$). Let $f_1 : H \to G_1$ and $f_2 : H \to G_2$ be any pair of homomorphisms. Then there exists a unique $h : H \to G_1 \oplus G_2$ such that $f_1 = g_1 \circ h$ and $f_2 = g_2 \circ h$.

- In other words, this diagram commutes

$$\begin{tikzcd}
H \arrow[dr, dotted, "\exists! h"] \arrow[drr, bend left, "f_2"] \arrow[ddr, bend right, "f_1"'] & & \\
& G_1 \oplus G_2 \arrow[r, "g_2"] \arrow[d, "g_1"'] & G2 \\
& G_1 &
\end{tikzcd}$$

Remarks: Before proceed we will introduce a important lemma in relation the initial object and make a claim of its dual.

Lemma 8. *Take an adjunction* $\mathscr{A} \underset{G}{\overset{F}{\rightleftarrows}} \mathscr{B}$ *and an object* $A \in \mathscr{A}$*. Then the unit map is an initial object of* $(A \Rightarrow G)$*.*

Lemma 9. *If* $\mathscr{A}$ *and* $\mathscr{B}$ *are categories and* $F : \mathscr{A} \to \mathscr{B}$ *and* $G : \mathscr{B} \to \mathscr{A}$ *be adjoint functors, then for* $B \in \mathscr{B}$*, the counit map* $\varepsilon_B : FG(B) \to B$ *is terminal in the comma category* $(F \Rightarrow B)$ *where* B *is the functor from the discrete category with one element to the category* $\mathscr{B}$*.*

It's obvious that **Lemma 9** is dual to **Lemma 8**. **Lemma 8** addresses the initial property of a given objective in the comma category in terms of unit. Conversely, **Lemma 9** addresses the terminal property of a given objective in the comma category in terms of counit. Generally, since most universal property can be derived via initial and terminal objects, **Lemma 8** and its dual are powerful tools to have. To prove the **Lemma 9**, we need to introduce another lemma:

Lemma 10. *Let* $\mathscr{A} \underset{G}{\overset{F}{\rightleftarrows}} \mathscr{B}$ *be an adjunction, with unit* η *and counit* ε*. Then* $\bar{g} = G(g) \circ \eta_A$ *for any* $g : F(A) \to B$*, and* $\bar{f} = \varepsilon_B \circ F(f)$ *for any* $f : A \to G(B)$*.*

Proof. Let $f : F(A) \to B$ be an object of $(F \Rightarrow B)$. We have to show there only exists one map from f to ε_B. A map $f \to \varepsilon_B$ in $(F \Rightarrow B)$ is a map $q : A \to G(B)$ in $\mathscr{A}$ such that:

$$\begin{tikzcd}
FG(B) \arrow[r, "\varepsilon_B"] & B \\
F(A) \arrow[u, "F(q)"] \arrow[ur, "f"'] &
\end{tikzcd}$$

commutes. But $\varepsilon_B \circ F(q) = \bar{q}$ by **Lemma 10**, so the diagram commutes if only if $f = \bar{q}$, and if only if $q = \bar{f}$. Hence, $F(q)$ is the unique map from $F(A)$ to $FG(B)$. $\square$

Having established the connection between universal property and terminal object, we will use such property to illustrate how to derive the universal property of direct product.

Proof. First, given cateories $Grp, \mathscr{D}$, and 1, a one object category, and functors F and B, we will construct the comma category $(F \Rightarrow B)$. We define define $\mathscr{D}$ as such:

- Objects: (G_1, G_2), a collection of two groups.

- Arrows: Let $(H_1, H_2), (G_1, G_2) \in \mathscr{D}$ be given, then $f : (G_1, G_2) \to (H_1, H_2)$ is pair of maps defined as follows:

  - $$\begin{tikzcd} & H_1 \\ G_2 \arrow[ur, "h_1"] \arrow[dr, "h2"'] & \\ & H_2 \end{tikzcd}$$

We define the functor F and B to be such:

- $F : \mathbf{Grp} \to \mathscr{D}$ is a functor.

  - $F : G \mapsto (1, G)$

- $B : 1 \to \mathscr{D}$ is another functor, where 1 is one object category.

  - $G : 1 \mapsto (G_1, G_2)$

- It implies the following diagram commutes:

$$\begin{tikzcd} & 1 \arrow[d, "B"] \\ Grp \arrow[r, "F"'] & \mathscr{D} \end{tikzcd}$$

We want to find a unique functor G that is adjoint to F. Consider the functor $G : \mathscr{D} \to Grp$, where it's defined as $G : (H, K) \mapsto H \oplus K$, a pair of group that is mapped to their direct product. By the **claim**, we know $\varepsilon_B : FG(B) \to B$ is terminal in the comma category $(F \Rightarrow B)$, given $B \in \mathscr{B}$, and in our set up of the comma category, $\varepsilon_{(G_1, G_2)} : (1, G_1 \oplus G_2) \mapsto (G_1.G_2)$ being terminal in $(F \Rightarrow B)$ precisely implies the following, given that $f : H \mapsto (G_1, G_2)$ is another object in this comma category:

- $$\begin{tikzcd}
 & & G_1 \\
H \arrow[rru, bend left, "f_1"] \arrow[r, "\exists! h"] \arrow[rrd, bend right, "f_2"'] & G_1 \oplus G_2 \arrow[ru, "\varepsilon_1"] \arrow[rd, "\varepsilon_2"'] & \\
 & & G_2
\end{tikzcd}$$

This commutative diagram is analogues to the universal property of direct product:

- $$\begin{tikzcd}
H \arrow[rrd, bend left, "f_2"] \arrow[rd, dotted, "\exists! h"] \arrow[rdd, bend right, "f_1"'] & & \\
 & G_1 \oplus G_2 \arrow[r, "g_2"] \arrow[d, "g_1"'] & G_2 \\
 & G_1 &
\end{tikzcd}$$

It implies that for any other pair of projection map, the arrow in the comma category, to be mapped to G_1, G_2. It has to factor through the direct product, as desired. □

3.4 Universal Property of Discrete Topology

In the study of topology, discrete topology is a topological space that all subsets are open. We will provide the universal property of it and then derive it from **Lemma 10**.

Definition 22. Given a set S, we can build a topological space $D(S)$ by equipping S with the discrete topology: all subsets are open. With this topology, any map from S to a space X is continuous.

- Define a function $i : S \to D(S)$ by $i(s) = s(s \in S)$. Then the following commutative diagram commutes:

  – $$\begin{tikzcd}
S \arrow[r, "i"] \arrow[rd, "\forall functions f"'] & D(S) \arrow[d, dotted, "\exists! continuous \bar{f}"] \\
 & \forall X
\end{tikzcd}$$

At this point, one should be familiar with the derivation of universal property. To reiterate, we start by construction the comma category, and then apply **Lemma 10** using the fact that the unit map is initial. However, in certain cases, it entails the fact that the counit maps are terminal. After building the commutative diagram of the unit or the counit map, the connection between the maps and the universal property becomes obvious.

Proof. First, given categories Top, Set, 1, the discrete category, and functors S, U, we will construct the comma category $(S \Rightarrow U)$. The categories Top and Set are the categories of topological spaces and sets respectivly. We define the functor S and U as such:

- U : **Top** $\to$ **Set** is a forgetful functor that forget the topological struture of a given set.
  - $U : T \mapsto S$, where $T \in$ **Top** and $S \in$ **Set**
- S : **1** $\to$ **Set**, where **1** is the one object category
  - $S : 1 \mapsto S$, $1 \in$ **1** and $S \in$ **Set**
- It implies the following commutative diagram commutes:

$$\begin{tikzcd} & Top \arrow[d, "U"] \\ 1 \arrow[r, "S"'] & Set \end{tikzcd}$$

We want to find a unique functor D that is adjoint to U. Consider the functor D : **Set** $\to$ **Top**, where it's defined as $D : S \mapsto D(S)$, where S is a set and $D(S)$ is the set equipped with discrete topology. Since U and D are forgetful functor and free functor respectively. so we know they are adjoint to each other. To show D is adjoint to U, we have to show that $\mathbf{Top}(D(S),T) \cong \mathbf{Set}(S,U(T))$. We define $g : D(S) \to T$ to be an arrow in **Top**, thus we have $\bar{g} : S \to U(T)$ as an arrow in **Set**. Let $q : T \to T'$ be a map in **Top**. Similarly, let $f : S \to U(T)$ to be an arrow in **Set**, then we have $\bar{f} : D(S) \to T$ as an arrow in **Top**. It suffices to show $q \bar{\circ} g = U(q) \circ \bar{g}$. Now we have $q \bar{\circ} g : S \to U(T')$ and $U(q) \circ \bar{g} : S \to U(T')$, so we haven shown that D and U are adjoint to each other. Thus, by **Lemma 10**, we know the the unit map η is initial in this comma category, and it implies that given $S \in \mathbf{Set}$, $\eta_S : S \mapsto UD(S)$ is initial in $(S \Rightarrow U)$. Note that $h : D(S) \mapsto T$ is the continous map between the discrete topological space to another arbitrary topological space T. Given another object of $(S \Rightarrow U)$, $f : S \mapsto U(T)$, η_S being initial implies the follwoing:

- $$\begin{tikzcd} S \arrow[r, "\eta_S"] \arrow[dr, "f"'] & UD(S) \arrow[d, dotted, "\exists! U(h)"] \\ & U(T) \end{tikzcd}$$

- This commutative diagram is analogous to the universal property of discrete topology:

$$\begin{tikzcd} S \arrow[r, "i"] \arrow[dr, "\forall functions f"'] & D(S) \arrow[d, dotted, "\exists! continuous \bar{f}"] \\ & \forall X \end{tikzcd}$$

It implies that for other projection maps from a set to a topological space, there exist a unique map from the discrete topological space to the other topological space.

Remarks: We should be aware of the abuse of notation in this case, indeed, $i : S \to D(S)$ means the functions between the sets S and $UD(S)$ that equips the set S with discrete topology. Even without the forgetful functor U : **Top** $\to$ **Set**, the maps between the underlying sets of their corresponding topologies are continous.

$\square$

3.5 Universal Property of Indiscrete Topology

The universal property of indiscrete topology is similar to the one of discrete topology. It's important to note that the indiscrete topology is dual to discrete topology, thus we would apply the dual of **Lemma 10** to derive the universal property via the terminal property of indiscrete topology. Consider the universal property of indiscrete topology:

Definition 23. Given a indiscrete topology $I(S)$, we could unequip the indiscrete topological struture of it, the only open sets are $I(S)$ and $\emptyset$ the set, to a set S.

- In other words, define a function $i : I(S) \to S$ by $i(I(s)) = s$, $s \in S$. X is some arbitrary topological space. Then the following commutative diagram commutes:

$$\begin{tikzcd} I(S) \arrow[r, "i"] & S \\ X \arrow[u, dotted, "\exists! h"] \arrow[ur, "f"'] & \end{tikzcd}$$

For any projection map between a topological space and its underlying set, there exists a unique map from the topological space and the indiscrete topological space.

Proof. First, given categories Top, Set, and 1, the discrete category, and functors U and S, we will construct the comma category $(U \Rightarrow S)$. We define U and S as such:

- U : **Top** $\to$ **Set** is a forgetful functor that forget the topological struture of a given set.
  - $U : T \mapsto S$, where $T \in$ **Top** and $S \in$ **Set**
- S : **1** $\to$ **Set**, where **1** is the one object category
  - $S : 1 \mapsto S$, $1 \in$ **1** and $S \in$ **Set**
- It implies the follwoing commutative diagram commutes:

$$\begin{tikzcd} & 1 \arrow[d, "S"] \\ Top \arrow[r, "U"'] & Set \end{tikzcd}$$

We want to find a unique functor I that is adjoint to U. Consider the free functor I : **Set** $\to$ **Top**, where it's defined by $I : s \mapsto I(s)$, where s is a set and $I(s)$ is the set equipped with indiscrete topology. Since U is the forgetful functor in relation to I as the free functor, so they are adjoint to each other. Thus, by the dual of **Lemma 10**, we know the counit map ε is terminal in this comma category. Given $S \in \mathbf{Set}$, the dual of **Lemma 10** implies $\varepsilon_s : UI(S) \to S$ is terminal in the comma category $(U \Rightarrow S)$. Note that $h : I(S) \to T$ is the continuous map between the indiscrete topological space, $I(S)$, and an aribitrary topological space T. The following commutative diagram represents an arrow of the comma category $(U \Rightarrow S)$, $f \mapsto \varepsilon_S$, and it commutes:

- $$\begin{tikzcd} UI(S) \arrow[r, "\varepsilon_S"] & S \\ U(T) \arrow[u, dotted, "\exists! U(h)"] \arrow[ur, "f"'] & \end{tikzcd}$$

- This commutative diagram is analogous to the universal property of indiscrete topology:

$$\begin{tikzcd} I(S) \arrow[r, "i"] & S \\ X \arrow[u, dotted, "\exists! h"] \arrow[ur, "f"'] & \end{tikzcd}$$

Remarks: The argument for the abusive notation is similar in the case of indiscrete topology. Without the forgetful functor U, the maps between two sets are still continuous.

$\square$

3.6 Summary

Overall this study, we begin by establishing the basic definitions of category, functor, natural transformation, and adjunctions. Providing relevant examples to achieve a better understanding of the topic. At the end, we start to derive universal properties for some mathematical concepts built upon the study of adjunctions. The universal properties proposed in this study are interpreted either in terms of initial object or in terms of terminal object, two fundamental concepts for the derivation. Universal property is an important direct application of category theory, but there are many more applications yet to be discussed.

References

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