How to Maximize the Profit for Bidder and Seller in a Sealed-Bid Second-Price Auction

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How to Maximize the Profit for Bidder and Seller in a Sealed-Bid Second-Price Auction

By

Wei Yu

Submitted in partial fulfillment of the requirements for Honors in the Department of Economics

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ABSTRACT


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With a history of more than 2500 years, auctions have long been used to negotiate the exchange of goods and commodities. In an auction, bidders compete with rivals by submitting bids depending on their personal evaluations of the goods. The good is allocated to the bidder who offers the highest bid. There are many different types of auctions, but four major ones are primarily concerned by economists and researchers--the English auction, the Dutch auction, the sealed-bid first-price auction and the sealed-bid second-price auction. My thesis mainly focuses on the characteristics of the sealed-bid second-price auction, with both continuous and discrete bidding. My thesis discusses the bidder's strategies that can maximum the expected payoff and the seller's strategy that can affect the expected revenue.

In continuous bidding, truthful bidding is the dominant strategy. In terms of the discrete bidding, my thesis applies the model from Yu (1999) to specifically discuss the sealed-bid second-price auction and finds out equilibrium strategy using the expected payoff function from buyers. My thesis discusses the trade-off between the winning probability and expected payoff for buyers and gives out suggestion on bidding based on individual's risk preference. Also, my thesis discusses the expected revenue function for sellers and the factors affecting the expected revenue.
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Chapter 1
Introduction

A. A Brief Review of Auction

The word "auction" is derived from the Latin *augeō* which means "I increase" or "I augment." The record of auction can be retrospect to early 500 B.C. where the women were arranged for their marriage. During the Romans period, auctions were used to liquidate the assets of debtors whose property had been confiscated. Nowadays, the main function of the auction is allocating and exchanging goods and services.

Most of the common auctions consist of one seller selling one or more goods and numbers of interested bidders. Bidders compete with their rivals by submitting personal bids depending on their personal valuations for the goods. The good is allocated to the bidder who offers the highest bid. However, the actual price paid by the winner does not always equal the highest bid, but depends on the auction type.

In regular market, sellers usually set prices of the good or service to be worthy of its property. However, sellers in auction seek the buyer whose personal valuation of the good or the service to be the highest. Therefore, though auctions do sell normal goods, they put more attention on art works, antiques, jewelries and other precious goods in which their values vary a lot among bidders. The auction can get a larger profit in the process of accepting increasingly higher biddings. Nowadays, bias of the auction market and regular market has been changed due

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1 Krishna, 2002: p2
to the development of the internet, which has led to a significant rise for the range of available buyers and categories of goods. The auction industry had a rapid growth recently. In 2008, the National Auctioneers Association reported that the gross revenue of the auction industry in the U.S. was approximately $268.4 billion. Nowadays, auctions have been applied in a wide range in society. The world's most famous wine auction, known as Hospice de Beaune, greatly promoted the reputation and the sale of famous wines. For example, France's former first lady, Carla Bruni, acted as the guest auctioneer in 2012 and a Ukrainian businessman offered Bruni and Sarkozy a bidding up to $350,000. The U.S. government also has treasury auction department for short and medium term government bonds. Even for online games, such as World of Warcraft, the game publisher Blizzard Entertainment, Inc. applies auction system to balance with monetary system within the game. Moreover, the E-commerce applies online auction system such as the internet auction site eBay.

Theoretically, researchers treat the auction as an incomplete information game. The exchange of good and the determination of price can be relatively easy to deal with due to the absence of market intervention. Namely, an auction can be treated as a sealed-single market in which the price is not affected by the outside market. Also, auctions can be treated as an incomplete information game since a bidder's strategic behavior is greatly affected by the information of his competitors' strategies. Hence, the game theory of incomplete information plays an important role in dealing with auctions. Applying game theory in auction helps bidders find out the optimal strategy for bidding so that they can maximize their profits.
B. Discrete Bid Auctions

Most of the existing literature about auctions focuses on the situation in which there are no restrictions on the bids. Generally speaking, a bidder is allowed to bid any arbitrary amount based on his or her valuation for the good. In other words, a bidder's choice is determined by his observed information and it is from a continuum of acceptable choices. However, in a real world auction, restrictions indeed exist. For example, the discrete nature of currency makes a restriction on the bidding choices. Another example would be the English auction in which the auctioneer sometimes sets a minimum price to reserve the value of the good, or restricts on the amount in which the next bid must be higher from the current bid. Considering the online auction site eBay, a bid increment will be imposed to the winner and this is also a form of restriction.

Mathematically, those bidding restrictions make the bidding space to be discrete. Compared to continuous bidding, even though bidders' valuations to the good are identically and independently distributed, they expected revenues may change. Furthermore, with discrete bidding space, the sealed-bid second-price auction no longer has a dominant strategy, which is the truthful bidding strategy under continuous bidding. This paper will discuss the characteristics of discrete bidding as well as the difference for strategic equilibrium between continuous and discrete bidding.
C. Purpose and Organization

My thesis mainly investigates the sealed-bid second-price auction, with discrete biddings. Also, my thesis discusses the equilibrium strategy where the profit for the bidder will be maximized. Secondly, my thesis applies Chwe's (1989) assumptions and Yu's (1999) model to seek an equilibrium strategy for buyers by conducting the expected payoff function. Finally, my thesis discusses the expected revenue for sellers and some of the influential factors that could affect the expected revenue.

Chapter 2 of my thesis is the literature review, which focuses on the games of incomplete information, auctions types and the discrete biddings. In Chapter 3, I discuss the literature of the sealed-bid second-price auction in continuous bidding space. In Chapter 4, I discuss the model for the sealed-bid second-price auction in discrete bidding space. In this case, my thesis discusses the expected payoff function for bidders and expected revenue function for sellers. In Chapter 5, I make the conclusion for my thesis and applications to the real world. Finally, the appendix contains the proofs of my thesis.
Chapter 2
Review of Existing Literature

This chapter reviews the existing literature about auction theory, which includes the game theory of incomplete information, auction types, the comparisons between different types of auctions and auctions with discrete bidding. The incomplete information game is one of the foundations of the auction theory. Also, the advantages and disadvantages among various type of auctions draw many attention from the researchers. Lastly, this chapter discusses the theoretical works that focus on the discrete bidding.

A. Games of Incomplete Information

In an auction, information is not fully symmetric among bidders since one can only know his own strategic behavior but not others'. We can treat an auction as an incomplete information game since the information is asymmetric in the auction. Hence, it is helpful to apply game theory of incomplete information to seek the equilibrium strategy in an auction.

Harsanyi’s (1967; 1968) studies are considered to be one of the great papers leading the modern information economics. Though it mainly focuses on game theory, its economic thought plays an important role. Information in reality is usually unevenly distributed and hence a Nash equilibrium cannot be easily achieved. Harsanyi (1968) considers the games with incomplete information where players lack some important parameters such as payoff functions, rival's
strategic behaviors and so on. The author sets up a new theory analyzing games with incomplete information and shows that, under certain assumption, such game can be equivalent to a certain game with complete information, called the "Bayes-equivalent" of the original game, or briefly a "Bayesian game." This study provides us a new way of achieving equilibrium in the games with incomplete information. Harsanyi (1973) also contributes a theorem solving the mixed strategy Nash equilibrium. Given that each player's personal information is not transparent to his rivals and only know to himself, the mixed strategy equilibrium can be explained as the limit of pure strategy equilibrium for a disturbed game of incomplete information. As approaches to the limit, the player's strategies converge to the predicted Nash equilibrium, which is equivalent to the equilibrium in the complete information game. Harsanyi (1968) provides us a fundamental support to find out the equilibrium in an auction game, which is considered as an incomplete information game. Harsanyi (1973) theory helps us to find out the equilibrium strategy in an auction game in which the bidders' strategies are mixed.

Akerlof (1970) discusses the information asymmetry between the seller and the buyer, and its influence to the market. The author uses the market for used cars as an example to illustrate the problem of quality uncertainty. The author points out that the asymmetric information provides the incentive for the seller to pass off low-quality goods as higher-quality ones and the buyer prefers to consider the average quality of the goods. This phenomenon is sometimes categorized as "the bad driving out the good" in the market. Therefore, such asymmetric information gives great influence on the market efficiency. Consider the auction market in which
the English auction has transparent information but sealed-bid auctions have asymmetric information, Akerlof's thought infers a great difference among auction types. The transparency of information certainly affects the efficiency of the auction market. In my study, I assume that each bidder only knows his or her own strategic behavior so that there is no second-thinker in the game. The result will be entirely different if all bidders know their rivals' strategic behaviors.

B. Auction Types and Comparisons

The institutional rule is the main factor to categorize auctions and it also has great influence on the bidding incentives, which has been mentioned by Vickrey (1961), of the bidders. There are many different types of auctions, but four major ones are primarily concerned by economists and researchers.

The first type of auction is called the English Auction, which is also known as the open ascending price auction. Bidders compete with rivals openly with each subsequent bid to be higher than the previous one. The good is allocated to the highest bidder and the price paid equals to the highest bid. Sometimes the seller will set a minimum price as a reserve price, and sometimes the seller will set a minimum amount requiring that the next bid much exceed such amount to the current bid. The most significant feature of English auction is that the current highest bid is always open to any potential bidders. This type of auction is arguably the most common form of auction in use today.
The second type of auction is called the Dutch Auction, which is also known as the *open descending price auction*. The auctioneer begins with a ceiling price and gradually lower it until some bidders are willing to accept. The good is allocated to the bidder who first accepts the price and the paid price is the last announced one. This type of auction is convenient when it is important to auction goods quickly, since a sale never requires more than one bid.

The third type of auction is called the Sealed-Bid First-Price Auction, which is also known as the *first-price sealed-bid auction* (FPSB). Bidders submit their personal sealed bids simultaneously without know others bidding information. The good is allocated to the highest bidder and the price paid equals the amount he or she submitted. Different from the English auction, bidders can only submit one bid. Also, the information is intransparent where bidders cannot change their bids according to their rivals' bids. This kind of auction is commonly used for government contracts and mining leases.

The fourth type of auction is called the Sealed-Bid Second-Price Auction, which is also known as the Vickrey auction. This auction is invented by Vickrey (1961) and named after him. Compared to the sealed-bid first-price auction, this is identical except the paid price by the winner equals to the second highest bid instead of the highest one. One example for this auction type would be the internet auction site eBay in which the auction system is almost identical to the sealed-bid second-price auction but an extra bidding increment. Besides these four major types, there are many secondary auction types such as all-pay auction, payout auction and so on.
There are indeed advantages and disadvantages for each of the four major types of auctions. The expected revenue, for example, is one of the most important factors that concerned by the sellers. Which kind of auction could offer the highest expected revenue so that it is the most favorable one to the sellers? It is surprising that, in a continuous auction, all four types of auction have the identical expected revenue for the seller. Under the assumption of risk neutrality, independence of private valuations and symmetry among bidders, Riley and Samuelson (1981) and Myerson (1981) show that all four types of auction will have the same expected revenue for the seller. Though this result is restricted in the equilibrium where the incentives to participate in the auction do not change, the four types of auction would be equivalent to the sellers in terms of expected revenue. Since the auction types are identical, in terms of the expected revenue, to the sellers, theoretical works put more attention on the expected payoff of the bidders as well as their strategic behaviors.

In reality, however, those mentioned assumptions--risk neutrality, independence of private valuations and symmetry among bidders-- may not be fulfilled. For example, in the case of natural resource auction, the assumption of independence of valuations fails. Therefore, a more general version of assumption requires the value of the good to bidders to be identical and the distribution function for each bidder's valuation is unbiased. Milgrom and Weber (1982) find out that the expected revenue is no longer identical among four types: the English auction provides a higher expected revenue than sealed-bid auction. Moreover, when the assumption changes again where the risk neutrality becomes risk aversion, expected revenue becomes higher in the
sealed-bid auctions. This result explains the fact that sealed-bid auctions are commonly used in offshore oil leases auctions because people in the auction are tend to be risk averse.

Compared to the English auction and the Dutch auction, researchers put more attentions on sealed-bid auctions nowadays. The Vickrey auction, which is also known as the sealed-bid second-price auction, has been first described in Vickrey (1961). Vickrey considers an auction where there is only one indivisible good is being sold and the paid price by the winner equals to the second highest bid. He finds that truthful bidding is the dominant strategy for each bidder regardless of bidders' risk attitudes. Since bidders in a Vickrey auction are likely to bid on their true valuations on the good, it is quite welcome by the sellers. For example, eBay's proxy bidding system is almost identical to the Vickrey auction except there is a bidding increment for each winner. Also, Google's and Yahoo!'s online advertisement system apply the Vickrey auction. Despite the strengths of Vickrey auction, it still has some shortages. Suppose, for example, bidders know the valuations of their rivals, they could lower their bid while preserving to win the good. Moreover, if the bidding level of bidders is restricted to a discrete bidding space, truthful bidding may not be a dominant strategy anymore.

C. Discrete Bidding.

Most of the existing theoretical works on auctions focus exclusively on the situation assuming there is no restrictions on the bid levels. This means that a bidder can bid any arbitrary amount on the good based on his personal valuation and observed information about his rivals.
However, restrictions on bids often exist in the real world. For example, the discrete nature of currency makes the acceptable bid levels restricted. The minimum amount of currency one can hold is, let us say, one cent. It is implausible for a bidder to bid lower than such amount. Also, for example, in an English auction, the seller sometimes sets a minimum price to preserve the value of the good, which is also a restriction on bids. Another example would be eBay, where the winner pays an extra bidding increment varies between five cents and one hundred dollars beside the bidding price. Therefore, restricted bidding deserves more attention because it reflects the reality.

Vickrey (1962) illustrates the situation in which each bidder only has two distinct bid levels. Under this assumption, the expected revenue for sellers are no longer identical among four types of auctions. Like Milgrom and Weber's (1982) result, the English auction has a higher expected revenue than that of the second-price auction. Though this kind of discrete bidding space is simple and unrealistic, it still gives us some hint about the difference by changing the continuous bidding space into a discrete one. In reality, the situation in which there are only two bid levels indeed exists. The Chinese government offers the contract for building a highway and the price is fixed as default. Bidder can only choose to accept or decline the offer. However, such case is relatively rare in reality. Hence, my thesis applies the model in which there are multiple of bid levels.

Chwe (1989) focuses on the sealed-bid first-price auction with discrete bidding where there are n bidders, each with an independent private valuation function and each bidder's valuation is
continuously distributed. The authors brings out the thought about "evenly spaced" discrete bid levels from a set \( M = \{b_1, b_2, ..., b_M\} \) such that \( b_i = \frac{i-1}{M} \). Based on the fact that overbidding is the dominant strategy in the sealed-bid first-price bidding, Chwe (1989) puts another bid \( b_{M+1} = \frac{(M+1)-1}{M} = 1 \) to make the maximum bid equals to the highest possible bidder valuation. He shows that there exists a unique symmetric Nash equilibrium bidding strategy and converges to the equilibrium of the continuous bidding auction when \( M \) approaches infinity. (i.e. the bid increment goes to 0) The authors argues that such auction has less revenue than the continuous bid auction but converges to the revenue in continuous bid auction. In reality, such evenly spaced bidding space is widely existing. In English auction and Dutch auction, the sellers usually increase or decrease the bid level by a certain amount, which makes the bidding space evenly distributed. In my thesis, the model applies the assumption that the bidding space is evenly distributed in order to mimic the real world situation.

Rothkopf and Harstad (1994) apply discrete bidding to an English auction. Quite interestingly, with discrete bid, an English auction is no longer strategically equivalent to a sealed-bid second-price auction where these two are equivalent with continuous bidding. The authors assume there are \( n \) bidders, each with an independent private valuation and \( m+1 \) discrete bidding levels. Under the situation where the bidder valuations are uniformly distributed, expected revenue of the seller can achieved at the cost of decreasing the expected difference between the highest bidder valuation and the valuation of the winner (which is also called the expected economic inefficiency). Also, the equilibrium is dynamic based on the number of
bidders and distribution of bidders' valuation functions. Those factors influencing the equilibrium provides a clue in my thesis about the determinant for the expected revenue for sellers. For example, the number of bidders indeed affects the expected revenue for sellers.

Yu (1999) applies Chwe's (1989) assumption about "evenly spaced" discrete values to examine each of the four primary auction types. Yu (1999) assumes that the valuations of the n bidders are independent from a common distribution function F(v) such that F(0) = 0 and F(1) = 1. In her paper, each type of the auctions has a symmetric pure strategy equilibrium. However, under such assumption, the truthful bidding is no longer the dominant strategy compared to overbidding and underbidding. This means some bidders will bid above their true valuation and some will bid below. Such phenomenon will lead to a market inefficiency. Also, Yu finds out that as the bidding increment goes to 0 (M goes to infinity), equilibrium converges to the equilibrium in continuous bid auction. Yu also assumes that the minimum bid level equals to 0 and the maximum bid level equals to 1. My thesis applies Yu's assumption because such assumptions simplify the model in terms of calculating the expected payoff for bidders.

Nowadays, online auctions often apply the discrete bidding space. Hence, researchers are trying to provide practical guidance as to how an auctioneer should determine the number and value of these discrete bid levels. David el at (2005) aim to provide the optimal auction design for an English auction with discrete bid levels. The author describe the discrete bidding space and to the end, derive an expression for the expected revenue of the seller. The expression is a function contains the actual discrete bid levels implemented, the number of bidders participating,
and the distribution from which the bidders draw their private independent valuations.

Specifically, comparing with previous theoretical work, the authors apply a uniform distribution to test their solutions. To conclude, the optimal bid levels result in improvements in the revenue, duration and allocative efficiency of the auction.

It is also worthy to mention another branch of discrete bidding in the literature. There is a phenomenon called “jump bidding” in ascending English auctions where bidders sometimes bid higher than what is necessary to be the current highest bidder. For example, Isaac et.al.(2007), Easley and Tenorio (2004) examine an auction form of open out-cry in English auctions. They also apply discrete bidding spaces that have special bidding restrictions. Such "jump bidding" discusses a case in which bidders have incentive to bid above his or her true value for the good. My study also mentions this phenomenon, which is correlated with the trade of between winning probability and expected payoff.
Chapter 3
The Sealed-Bid Second-Price Auction with Continuous Bidding

In this chapter, we discuss the sealed-bid second-price auction in which its bidding space is continuous and there is only one single, indivisible good is being sold. The sealed-bid second-price auction is also known as the Vickrey auction that was initially described by Professor William Vickrey in 1961. The sealed-bid second-price auction provides bidders an incentive to bid on their true value. In other word, truthful bidding is the dominant strategy for every bidders.

To claim that truthful bidding is the dominant strategy, we provide the following statement.

Statement:

The dominant strategy in a sealed-bid second-price auction with continuous biddings for a single, indivisible good is for each bidder to bid on his or her true value of the good.

Proof for the statement:\(^2\)

Let \( v_i \) be bidder \( i \)'s value for the good. Let \( b_i \) be bidder \( i \)'s bid for the good. The payoff \( \pi \) for bidder \( i \) is

\[
\pi = \begin{cases} 
  v_i - \max_{j \neq i} b_j & \text{if } b_i > \max_{j \neq i} b_j \\
  0 & \text{otherwise}
\end{cases}
\]

\(^2\) Similar proof can be found at Riley and Samuelson (1981) and Myerson (1981)
Now we are going to discuss that truthful bidding has a higher expected payoff than either overbidding and underbidding.

Case 1: Overbidding

Assume that bidder $i$ bids $b_i > v_i$, which means bidder $i$’s bid exceed his value of the good.

If $v_i > \max_{j \neq i} b_j$, then the bidder $i$ will win the good since $b_i > v_i > \max_{j \neq i} b_j$ with overbidding. Under the circumstance of a truthful bid where $b_i = v_i$, the bidder $i$ still wins the good since $b_i = v_i > \max_{j \neq i} b_j$. The payoff of bidder $i$ is independent of $b_j$ so that these two strategies have equal payoffs in this case.

If $b_i < \max_{j \neq i} b_j$, then the bidder $i$ will lose the good, both in overbidding and truthful bidding. They payoffs of both strategies will equally be 0.

If $v_i < \max_{j \neq i} b_i < b_i$, then the bidder $i$ will win the good under overbidding but with a negative payoff since $\pi = v_i - \max_{j \neq i} b_j < 0$. Under truthful bidding, the bidder $i$ will lose the good since $v_i = b_i < \max_{j \neq i} b_j$ and hence the payoff is 0.

To conclude, the strategy of overbidding is dominated by the strategy of truthful bidding in terms of the payoff function.

Case 2: Underbidding

Assume that bidder $i$ bids $b_i < v_i$, which means his bid is less than his value of the good.
If \( v_i < \max_{j \neq i} b_j \), then the bidder \( i \) will lose the good since \( b_i < v_i < \max_{j \neq i} b_j \). Under truthful bidding, the bidder will also lose the good since \( b_i = v_i < \max_{j \neq i} b_j \). These two strategies offer the same payoff to be 0 in this case.

If \( \max_{j \neq i} b_j < b_i \), then the bidder \( i \) will win the good either in underbidding or truthful bidding since his bid is the highest. Again, the payoff of bidder \( i \) is independent of \( b_i \) so that these two strategies have equal payoffs in this case.

If \( b_i < \max_{j \neq i} b_j < v_i \), then the bidder \( i \) will lose the good in underbidding and his payoff will be 0. However, under truthful bidding, the bidder \( i \) will win the good since \( \max_{j \neq i} b_j < v_i = b_i \). Also, they payoff \( \pi = v_i - \max_{j \neq i} b_j > 0 \) is positive.

To conclude, the strategy of underbidding is dominated by the strategy of truthful bidding.

In both cases, truthful bidding dominates the other possible strategy. Therefore, in the sealed-bid second-price auction with continuous bidding, truthful bidding is an optimal strategy.

To conclude in this case, the sealed-bid second-price auction is efficient since it provides incentive for bidders to bid on their true value of the good. In terms of the bidder, bidding on their true value grants them a non-negative payoff. This is because, by bidding on true value of the good, the bidder will either win the good getting a non-negative payoff, or lose the good getting a zero payoff. In terms of the seller, the truthful bidding strategy offers the seller's highest
expected revenue. These results have been showed in Riley and Samuelson (1981) and Myerson (1981).
Chapter 4  
Model for Sealed-Bid Second-Price Auction with Discrete Bid

In this chapter, we will introduce the model describing the sealed-bid second-price auction with discrete bidding space. In part A, we will introduce the basic set up of the model, including the assumptions, the basic structure and notations for the model. In part B, we will discuss the strategic behavior for a bidder, given his valuation of the good. In part C, we give out a fundamental concept about the equilibrium point in bidding levels, which is important in the following part. In part D, we apply the expected payoff function for bidders to discuss the equilibrium strategies. In part E, we discuss the expected revenue for sellers.

A. The Basic Set up of the Model for Sealed-Bid Second-Price Auction with Discrete Bidding

There are researchers who have built models for sealed-bid second-price auction have been studied by several researchers, such as Vickrey (1961), Chwe (1989) and Yu (1999). In my thesis, we are inspired by Chwe's (1989) assumptions and manipulate Yu's (1999) model for the sealed-bid second-price auction. The actual set up for the model is described as follows.
This model discusses an auction in which the seller auctions an object to \( N \) bidders. Each bidder's value of the object is distributed independently with \( v_i \in [v_L, v_H] = [0, 1] \), with cumulative continuous distribution function \( F(v) \) such that \( F(0) = 0 \) and \( F(1) = 1 \). Each bidder only knows his own value of the object and bids \( b_i \) from a set containing \( M+1 \) discrete bid levels \( B = \{b_1, b_2, ..., b_{M+1}\} \) where \( b_i = \frac{i-1}{M} \). Assume that \( b_1 = 0 \) and \( b_{M+1} \leq 1 \). This means the bid possibilities are multiples of the increment \( \frac{1}{M} \). Follow the requirement of the seal-bid second-price auction, the highest bidder receives the good paying a price equal to the second highest bid. If more than one bidders bid the same highest \( b_i \), then randomly and fairly select one of them to receive the good.

**B. The Strategic Behavior for Bidders**

Here we are going to discuss the strategic behavior for an arbitrary bidder given his or her personal value for the good. The strategic behavior in auction markets describes the preference of bidding for a certain bidder.

Suppose there are finite number of discrete bid levels. There exists at least an acceptable bid level. This following Proposition 1 is inspired from Mathews' (2008) Proposition 1.

**Proposition 1:**

---

\(^3\) This follows Chwe's (1989) and Yu's (1999) assumptions.
Consider an arbitrary acceptable bid level $b_j$. For bidder $i$ whose value $v_i \leq b_j$, bidding $b_j$ weakly dominates bidding above $b_j$; for bidder $i$ whose value $v_i \geq b_j$, bidding $b_j$ weakly dominates bidding below $b_j$.

Consider a bidder whose value $v_i \in (b_{j-1}, b_j)$. Proposition 1 implies that all bids other than $b_{j-1}$ and $b_j$ are weakly dominated. From Proposition 1, we have

1. A bidder with $v_i \geq b_j$ will have higher payoff from bidding $b_j$ than from bid levels below $b_j$.
2. A bidder with $v_i \leq b_j$ will have higher payoff from bidding $b_j$ than from bid levels above $b_j$.

Proposition 1 has restricted the plausible bid levels to $b_{j-1}$ and $b_j$, given that the value for a bidder $v_i \in (b_{j-1}, b_j)$. This helps us to narrow down the range of bid levels that are could be discussed.

C. Equilibrium Point in Bidding Levels

In this part, we will discuss that there exists an equilibrium point, or so-called the "mid-point", between any two consecutive bid levels $b_{j-1}$ and $b_j$ that it is identical to bid on $b_{j-1}$ or $b_j$ in terms of the expected payoff for bidders.
First consider bidder $i$ with $v_i \in (b_{j-1}, b_j)$ from bidding either $b_{j-1}$ or $b_j$. Let a two-tuple $E\pi(v_i, b_j)$ denote the expected payoff for bidder $i$, where $v_i$ is bidder $i$'s value of the good and $b_j$ is the bid he offers. The expected payoff for bidder $i$ from bidding $b_j$ is

$$E\pi(v_i, b_j) = \sum_{t=1}^{i-1} (v_i - b_t) p_t + \sum_{k=1}^{N} \frac{1}{k} (v_i - b_j) p_j$$  \hspace{1cm} (3.1)$$

where $k$ stands for the number of bidders who bid on $b_j$ and $p_i$ is the conditional probability that the second highest bid equals to $b_j$.

The description for the expected payoff function is as follows. The two-tuple $E\pi(v_i, b_j)$ represents the expected payoff for a bidder whose value for the good is $v_i$ and bids on $b_j$. Recall that in the sealed-bid second-price auction, the actual paid price equals to the second highest bid instead of the highest one. Then the first term of the equation (1), $\sum_{t=1}^{i-1} (v_i - b_t) p_t$, describes the expected payoff in which the second highest bid is less than $b_j$. The second term of the equation (1), $\sum_{k=1}^{N} \frac{1}{k} (v_i - b_j) p_j$, indicates the situation in which there are more than one bidders bid on the highest bid $b_j$. In this case, the winner is chosen randomly and fairly.

For the sake of simplifying the question, we first look $N = 2$. So the expected payoff function would be

$$E\pi(v_i, b_j) = \sum_{t=1}^{i-1} (v_i - b_t) p_t + \frac{1}{2} (v_i - b_j) p_j$$  \hspace{1cm} (3.2)$$

The difference between the expected payoff from bidding $b_{j-1}$ and $b_j$ is
Consider the situation $v_i = v_L = 0$. We have

\[
E\pi(0, b_j) = -\sum_{t=1}^{j-1} b_t p_t - \frac{1}{2} b_j p_j
\]

and

\[
E\pi(0, b_{j-1}) = -\sum_{t=1}^{j-2} b_t p_t - \frac{1}{2} b_{j-1} p_{j-1}
\]

We have that

\[
E\pi(0, b_j) < E\pi(0, b_{j-1}) \tag{3.4}
\]

This means that for bidder whose personal value for the good is 0, he would like to bid as low as possible.

Then we consider the partial derivative of the expected payoff function in terms of $v_i$

\[
\frac{\partial E\pi(v_i, b_j)}{\partial v_i} = \sum_{t=1}^{j-1} p_t + \frac{1}{2} p_j
\]

and

\[
\frac{\partial E\pi(v_i, b_{j-1})}{\partial v_i} = \sum_{t=1}^{j-2} p_t + \frac{1}{2} p_{j-1} = \sum_{t=1}^{j-1} p_t - \frac{1}{2} p_{j-1}
\]

Since $p_j > 0$ for all $j$, we have

\[
\frac{\partial E\pi(v_i, b_{j-1})}{\partial v_i} < \frac{\partial E\pi(v_i, b_j)}{\partial v_i} \tag{3.5}
\]

Equation (3.5) tells us that the expected revenue with a higher bidding is more sensitive to the change of personal value for the good.
Consider the situation that bidder's value for the good falls exactly on \( b_j, \ v_i = b_j \).

Substitute \( v_i = b_j \) in (3), we get

\[
E\pi(b_j, b_j) - E\pi(b_j, b_{j-1}) = \frac{1}{2} (b_j - b_j)p_j + \frac{1}{2} (b_j - b_{j-1})p_{j-1} > 0
\]

So we have

\[
E\pi(b_j, b_j) > E\pi(b_j, b_{j-1})
\]

(3.6)

Equation (3.6) illustrates the same property as Proposition 1-(1), which says that a bidder with \( v_i \geq b_{j-1} \) will have higher payoff from bidding \( b_{j-1} \) than from bid levels below \( b_{j-1} \).

Similarly, consider \( v_i = b_{j-1} \). Substitute \( v_i = b_{j-1} \) in (3.3), we get

\[
E\pi(b_{j-1}, b_{j-1}) - E\pi(b_{j-1}, b_{j-1}) = \frac{1}{2} (b_{j-1} - b_{j-1})p_j + \frac{1}{2} (b_{j-1} - b_{j-1})p_{j-1} < 0
\]

So we have

\[
E\pi(b_{j-1}, b_{j-1}) < E\pi(b_{j-1}, b_{j-1})
\]

(3.7)

Equation (3.7) illustrates the same property as Proposition 1-(2), which says that a bidder with \( v_i \leq b_{j-1} \) will have higher payoff from bidding \( b_{j-1} \) than from bid levels above \( b_{j-1} \).

Now, consider equation (3.6) \( E\pi(b_j, b_j) > E\pi(b_j, b_{j-1}) \) and equation (3.7)

\( E\pi(b_{j-1}, b_{j-1}) < E\pi(b_{j-1}, b_{j-1}) \). Since the expected payoff function is linear, then the Intermediate-Value Theorem suggests that there exist an unique \( s_j \in (b_{j-1}, b_j) \) such that

\[ E\pi(s_j, b_{j-1}) = E\pi(s_j, b_j). \]

To generalize this, let \( j = k \). We have that there exists an unique \( s_k \in (b_{k-1}, b_k) \) for all \( b_i \in B \). This is also true for \( s_k \in (v_H, b_k) = (1, b_k) \).
Here we call such $s_j$ to be the equilibrium point between $b_{j-1}$ and $b_j$ since it the expected payoff for bidding on $b_{j-1}$ and $b_j$ are the same, given that bidder's value for the good is $s_j$. We can treat $s_j$ to be the "mid-point" for the expected payoff between $b_{j-1}$ and $b_j$.

In Proposition 1, we have shown that bidder $i$ with $v_i \in (b_{j-1}, b_j)$ will only bid $b_{j-1}$ or $b_j$ since other bid levels are weakly dominated. Since $v_i > b_{j-1}$, then bidder $i$ will get a positive payoff by bidding $b_{j-1}$. Consider $b_j$, which is greater than $v_i$. In a sealed-bid second-price auction, a bidder $i$ with $v_i$ could still bid $b_j$ because the actual price he pays is equal to the second highest price instead of $b_j$. By bidding $b_j$, the bidder $i$ increases the corresponding winning probability by suffering a risk of getting negative payoff.

For any bidder with $v_i \in (b_{j-1}, b_j)$, we know that $s_j \in (b_{j-1}, b_j)$ gives a equilibrium point at which the expected payoff from bidding $b_{j-1}$ and $b_j$ are equal. If $v_i \in [b_{j-1}, s_j)$, then bidding $b_{j-1}$ will offer a higher payoff than bidding $b_j$. If $v_i \in (s_j, b_j]$, then bidding $b_j$ will offer a higher payoff than bidding $b_{j-1}$.

Recall that we assume the set of bid levels contains $M+1$ discrete points, $B = \{b_1, b_2, \ldots, b_{M+1}\}$. Then for any $(b_{j-1}, b_j)$, there exists a $s_j \in (b_{j-1}, b_j)$ such that $E\pi(s_j, b_{j-1}) < E\pi(v, b_j)$. Let $S = \{s_0, s_1, s_2, \ldots, s_r\}$ such that $0 = v_L = s_0 < s_1 < s_2 < \ldots < s_r = v_H = 1$ with $r \leq M + 1$.

Therefore, a bidder $i$ with $v_i \in [s_{j-1}, s_j)$ will bid $b_i$ since $v_i < s_j$ and a bidder with $v_i = s_r$ will bid $b_r$. Those explanations provide us a clear incentive to construct the expected payoff function for bidders in the following part.
D. The Equilibrium Strategy in Sealed-Bid Second-Price Auction

In this part, we will construct a function describing the optimal strategy for a bidder in terms of his or her expected payoff. Here we first discuss the strategy function for a bidder.

A bidder's strategy is a function from \([0,1]\) to \(B = \{b_1, b_2, ..., b_{M+1}\}\) returning the bidders' optimal \(b_i\), given the bidder's personal value \(v\) of the object. Denote this function as \(b(v) : [0,1] \rightarrow \{b_1, b_2, ..., b_{M+1}\}\).

A bidder's strategy \(b(v)\) is an equilibrium strategy if it satisfies

\[
E\pi(v, b_i) \geq E\pi(v, b_j), \forall b_j \in B = \{b_1, b_2, ..., b_{M+1}\}
\]  
(4.1)

and

\[
E\pi(v, b_i) \geq 0
\]  
(4.2)

Basically these two inequalities give out the restrictions of being an equilibrium. Namely, (4.1) infers that an equilibrium strategy offers the highest expected payoff among all strategies, and (4.2) infers that an equilibrium strategy should not let the bidder lose profit.

Recall that in a sealed-bid second-price auction, the good is allocated to the bidder who offers the highest bid and the actual payment is equal to the second highest bid. Let \(P_i\) denote the payment that a bidder bidding \(b_i\) needs to pay. Denote "the second highest bid" as SHB and "the highest bid" as HB. We have the expected payoff function for bidding \(b_i\):
\[ E\pi(v, b) = (v - P)b \text{Prob}(b \text{ is the HB|bidding } b) \]

The above function literally tells us that the expected payoff is the difference between one's personal value to the good and the actual paid price, times the corresponding probability.

A game in normal form\(^4\) is symmetric if all agents have the same strategy set, and the payoff is depends on the given strategy, not on the agents. Hence, an equilibrium strategy in a symmetric game is called a symmetric equilibrium strategy. The following four conditions allow us to construct the form of the symmetric equilibrium strategy \( b(v) \) (proofs for condition b, c, d can be found in Appendix)\(^5\).

a) \( b(0) = 0 \) It is easy to show that, under truthful bidding, a bidder's optimal bid will be 0 given that his value of the good is 0.

b) \( B = \{v \in [0,1] | b(v) = b_1 \} \) is convex in \( v \).

c) \( b(v) = 0, \forall v \in [0,1] \) is not an equilibrium strategy

d) \( b(v) \) is monotonically increasing in \( v \).

The symmetric equilibrium strategy \( b(v) \) we considered will be of the following form:

\[
b(v) = \begin{cases} 
  b_i & \text{if } v \in [s_{i-1}, s_i), 1 \leq i \leq r \\
  b_r & \text{if } v = s_r 
\end{cases} \tag{4.3}
\]

where \( S = \{s_0, s_1, s_2, ..., s_r \} \) is the strategy space for the bidder with

\[
0 = v_L = s_0 < s_1 < s_2 < ... < s_r = v_H = 1 \quad \text{and} \quad r \quad \text{is an integer such that} \quad 1 \leq r \leq M + 1. \quad \text{Equation (4.3)}
\]
can be explained into two separate cases. For bidder whose personal value is not as high as 1,

\(^4\) Adapted from Definition 7.D.2 in Mas-Colell et al. (1995)
\(^5\) Conditions b, c, d are the Lemma 1, 2, 3, respectively, in Yu's (1999) page 20.
then there exists a closest "mid-point" to the bidder's personal value. In this case, the bidder should choose the bid level having the same subscript with that "mid-point". For bidder whose personal value is 1, the bidder should just bid 1. We can show that the strategy form (4.3) is a symmetric (pure strategy) Nash equilibrium.

The existence of such pure strategy equilibrium has been discussed in many theoretical works. From the Purification Theorem in Milgrom and Weber (1985), there exists a symmetric equilibrium strategy in any finite symmetrical strategic-form game. Yu (1999) has discussed that all the four major types of auction are symmetrical games and she proved the existence of a symmetric pure strategy equilibrium.\(^6\)

Recall that the bidding strategy \( b(v) \) is an equilibrium strategy if equation (4.1) and (4.2) are satisfied at the same time. Note that in the sealed-bid second-price auction, the expected payoff for any bidder would be greater than 0. Therefore, equation (4.2) will be satisfied. i.e.

\[
E\pi(v, b_r) \geq 0, \forall v \in [0,1]
\]

Suppose a bidder bid \( b_r \) instead of \( b_{r+1} \), then it must be either \( E\pi(v, b_r) \geq E\pi(v, b_{r+1}) \) or \( b_{r+1} \) is not available. i.e. \( b_r = 1 \) and \( r = M + 1 \). Now we introduce two lemmas\(^7\) that are important in later proofs.

**Lemma 1\(^8\):**

\(^6\) Yu (1999) proved the existence in Proposition 1.
\(^7\) Lemma 1&2 are Lemma 4&5 in Yu (1999)
\(^8\) Proof for Lemma 1 is in Yu (1999) page 55.
Given that $E\pi(v, b_i) \geq E\pi(v, b_j), \forall b_j \in B = \{b_1, b_2, ..., b_{M+1}\}$ for each $i = 1, 2, ..., r - 1$, a bidder with value $s_j$ is indifferent from bidding $b_i$ and $b_{i+1}$, i.e. $E\pi(s_j, b_j) = E\pi(s_j, b_{i+1})$.

Lemma 1 is basically follows the definition of $S = \{s_0, s_1, s_2, ..., s_r\}$. It tells us that if the bidder's personal value falls exactly on some "mid-point," then the expected payoff would be the same from bidding above or below that "mid-point."

**Lemma 2**:  
\[
E\pi(s_r, b_r) \geq E\pi(s_r, b_{r+1}) \tag{4.4}
\]
for $1 \leq r \leq M + 1$, and,
\[
E\pi(s_i, b_i) = E\pi(s_i, b_{i+1}) \tag{4.5}
\]
for $i = 1, 2, ..., r - 1$ are the only binding constraints.

With these two lemmas, we can then determine that the bidding strategy $b(v)$ is an equilibrium strategy, stated in the following Proposition 2.

**Proposition 2**:  
The bidding strategy $b(v)$ of the form (4.3) is an equilibrium strategy if
\[
E\pi(s_r, b_r) \geq E\pi(s_r, b_{r+1})
\]
for $1 \leq r \leq M + 1$ and
\[
E\pi(s_i, b_i) = E\pi(s_i, b_{i+1})
\]
for $i = 1, 2, ..., r - 1$.

---

9 Proof of Lemma 2 is in Yu (1999) page 55.
10 Proof of Lemma 2 is in Yu (1999) page 56.
Recall that in the sealed-bid second-price auction, each bidder submits a sealed bid to the seller by his own value of the good. The good will be allocated to the bidder who bids the highest, and the price he pays equals to the second highest bid.

We have the expected payoff function in the following form:

\[ E\pi(v, b_i) = (v - P_i) \text{Prob}(b_i \text{ is the HB bidding } b_j) \]

Given that the bidder \( i \) wins the good by bidding \( b_i \), we need to calculate the value of \( P_i \), which stands for the actual price the bidder \( i \) needs to pay and is determined by the second highest bid among bidders. We can calculate the probability of each \( b_i \) to be the second highest bid and its corresponding payoff. Then we can get the expected payoff function by summing the calculated payoffs. Suppose that in the case of tie, the winner will be chosen randomly and fairly.

So we expand the expected payoff function as follows:

\[
E\pi(v, b_i) = (v - b_i) \text{Prob}(SHB = b_1 = 0 \mid HB=b_i) \\
+ (v - b_2) \text{Prob}(SHB = b_2 \mid HB=b_i) \\
+ ... \\
+ (v - b_{i-1}) \text{Prob}(SHB = b_{i-1} \mid HB=b_i) \\
+ (v - b_i) \sum_{t=2}^{N} \text{Prob}(t \geq 2 \text{ bidders bid } b_i \mid HB=b_i)
\]

Also, we have the expected payoff function for bidding \( b_{i+1} \):
\[
E\pi(v, b_{t+1}) = (v - b_1)\text{Prob}(SHB = b_1 = 0 \mid HB = b_{t+1})
+ (v - b_2)\text{Prob}(SHB = b_2 \mid HB = b_{t+1})
+ \ldots
+ (v - b_{t-1})\text{Prob}(SHB = b_{t-1} \mid HB = b_{t+1})
+ (v - b_t)\sum_{t=1}^{N} \text{Prob}(t \geq 2 \text{ bidders bid } b_{t+1} \mid HB = b_{t+1})
\]

(4.7)

Recall from the proposition 1 that \( h(v) \) is an equilibrium strategy if it satisfies condition (4.4): \( E\pi(s_i, b_r) \geq E\pi(s_r, b_{r+1}) \) for \( i = r \), and (4.5): \( E\pi(s_i, b_i) = E\pi(s_i, b_{i+1}) \) for \( i = 1, 2, \ldots, r - 1 \) are the only binding constraints. Plug (4.6) and (4.7) into condition (4.5), we have that:

\[
(s_i - b_j)\text{prob}(\text{SHB}=b_j = 0 \mid \text{HB} = b_i) = (s_i - b_j)\text{prob}(\text{SHB}=b_j = 0 \mid \text{HB} = b_{i+1})
\]

for \( k = 1, 2, \ldots, r - 1 \).

Hence, (4.7) - (4.6) will get:

\[
(s_i - b_{i+1})\sum_{t=1}^{N} \frac{1}{t} \text{Prob}(t \geq 2 \text{ bidders bid } b_{i+1} \mid \text{HB} = b_{i+1})
+ (s_i - b_j)\text{Prob}(\text{SHB}=b_j \mid \text{HB} = b_{i+1})
\]

(4.8)

\[
- (s_i - b_j)\sum_{t=2}^{N} \frac{1}{t} \text{Prob}(t \geq 2 \text{ bidders bid } b_j \mid \text{HB} = b_j)
\]

\[
= 0
\]

The reason why we combine equation (4.6) and (4.7) is that we can cancel many similar terms based on the fact that the expected payoff by bidding on \( b_i \) and \( b_{i+1} \) are same, given that the personal value for the good falls exactly on \( s_i \).

Now we are going to discuss the conditional probability presented in the equation above.

For \( \text{Prob}(t \geq 2 \text{ bidders bid } b_{t+1} \mid \text{HB}=b_{t+1}) \). Given that the highest bid is \( b_{t+1} \) and there are \( t \geq 2 \) bidders bid \( b_{t+1} \). Then besides the winner, there are \( t - 1 \) bidders bidding \( b_{t+1} \) among
\(N - 1\) bidders. Since \(s_i < b_i < s_{i+1}\), then the probability for those bidders with value \(b_i\) would be \((F(s_{i+1}) - F(s_i))^{-1}\). For the rest \((N - 1) - (t - 1) = N - t\) bidders, their bid levels are less than \(s_i\). So the probability is \((F(s_i))^{N-t}\). To conclude, we have that

\[
\text{Prob}(t \geq 2 \text{ bidders bid } b_{i+1} \mid \text{HB}=b_{i+1}) = \left(\binom{N-1}{t-1}(F(s_{i+1}) - F(s_i))^{-1}(F(s_i))^{N-t}\right)
\]

For \(\text{Prob}(\text{SHB} = b_i \mid \text{HB} = b_{i+1})\). Given that the highest bid is \(b_{i+1}\), then for the rest \(N - 1\) bidders, their probability is \(F^{N-1}(s_i)\). Given the second highest bid is \(b_i\), then for the rest \(N - 1\) bidders, their probability is \(F^{N-1}(s_{i-1})\). Therefore, we have

\[
\text{Prob}(\text{SHB} = b_i \mid \text{HB} = b_{i+1}) = F^{N-1}(s_i) - F^{N-1}(s_{i-1})
\]

For \(\text{Prob}(t \geq 2 \text{ bidders bid } b_i \mid \text{HB}=b_i)\). Given that the highest bid is \(b_i\) and there are \(t \geq 2\) bidders bid \(b_i\). Then besides the winner, there are \(t - 1\) bidders bidding \(b_i\) among \(N - 1\) bidders. Since \(s_{i-1} < b_i < s_i\), then the probability for those bidders with value \(b_i\) would be \((F(s_i) - F(s_{i-1}))^{-1}\). For the rest \((N - 1) - (t - 1) = N - t\) bidders, their bid levels are less than \(s_{i-1}\). So the probability is \((F(s_{i-1}))^{N-t}\). To conclude, we have that

\[
\text{Prob}(t \geq 2 \text{ bidders bid } b_i \mid \text{HB}=b_i) = \left(\binom{N-1}{t-1}(F(s_i) - F(s_{i-1}))^{-1}(F(s_i))^{N-t}\right)
\]

Substitute these probabilities into equation (8) we get

\[
(s_i - b_i) \sum_{i=2}^{N} \frac{1}{t} \left(\binom{N-1}{t-1}(F(s_i) - F(s_{i-1}))^{-1}(F(s_i))^{N-t}\right)
+ (s_i - b_i) \left(F^{N-1}(s_i) - F^{N-1}(s_{i-1})\right)
- (s_i - b_i) \sum_{i=2}^{N} \frac{1}{t} \left(\binom{N-1}{t-1}(F(s_i) - F(s_{i-1}))^{-1}(F(s_i))^{N-t}\right)
= 0
\]
Equation (4.9) contains the conditional probability so that we can derive an equation based on the probability density function. The following steps are the process of solving (4.9).

Notice that

\[
\binom{N}{t} = \binom{N-1}{t-1}^N
\]

and

\[
a^N = (a-b+b)^N = \sum_{t=0}^{N} \binom{N}{t} (a-b)^t b^{N-t}
\]

We have that

\[
\sum_{t=2}^{N} \frac{1}{t} \binom{N-1}{t-1} (a-b)^{t-1} b^{N-t} = \sum_{t=2}^{N} \frac{1}{t} \binom{N}{t} (a-b)^t b^{N-t} = \frac{N(a-b)}{N(a-b)}
\]

\[
= \left( \sum_{t=0}^{N} \frac{1}{t} \binom{N}{t} (a-b)^t b^{N-t} \right) - \binom{N}{0} b^N - \binom{N}{1} (a-b)^1 b^{N-1}
\]

\[
= \frac{a^N - b^N - N (a-b)^1 b^{N-1}}{N(a-b)}
\]

Hence, equation (4.8) can be rewritten as
\[(s_i - b_{t+1}) \sum_{t=1}^{N} \frac{1}{t} \left( F(s_t) - F(s_{t-1}) \right) \left( F(s_{t-1}) \right)^{N-t} \]
\[+ \ (s_i - b_j) \left( F^{N-1}(s_i) - F^{N-1}(s_{t-1}) \right) \]
\[- \ (s_i - b_j) \sum_{t=1}^{N} \frac{1}{t} \left( F(s_t) - F(s_{t-1}) \right) \left( F(s_{t-1}) \right)^{t-1} \left( F(s_{t-1}) \right)^{N-t} \]
\[= \ (s_i - b_{t+1}) \left( \frac{F^N(s_{t+1}) - F^N(s_i)}{N(F(s_{t+1}) - F(s_i))} - F^{N-1}(s_i) \right) \]
\[+ \ (s_i - b_j) \left( \frac{F^{N-1}(s_j) - F^{N-1}(s_{t-1})}{N(F(s_j) - F(s_{t-1}))} \right) \]
\[- \ (s_i - b_{t+1}) \left( \frac{F^N(s_i) - F^N(s_{t+1})}{N(F(s_i) - F(s_{t+1}))} - F^{N-1}(s_i) \right) \]
\[= \ (s_i - b_{t+1}) \left( \frac{F^N(s_{t+1}) - F^N(s_i)}{N(F(s_{t+1}) - F(s_i))} - F^{N-1}(s_i) \right) \]
\[+ \ (s_i - b_j) \left( \frac{F^{N-1}(s_j) - F^{N-1}(s_{t-1})}{N(F(s_j) - F(s_{t-1}))} \right) \]
\[= \ 0 \] (4.10)

From Yu (1999)\(^{11}\), we have the following Lemma 3 and the proof is in Appendix.

**Lemma 3**

\[\frac{F^N(s_i) - F^N(s_{t-1})}{N(F(s_i) - F(s_{t-1}))} < F^{N-1}(s_i) < \frac{F^N(s_{t+1}) - F^N(s_i)}{N(F(s_{t+1}) - F(s_i))}\]

From Lemma 3, we simplify equation (4.9)

\[(s_i - b_{t+1}) \left( \frac{F^N(s_{t+1}) - F^N(s_i)}{N(F(s_{t+1}) - F(s_i))} - F^{N-1}(s_i) \right) \]
\[+ \ (s_i - b_j) \left( \frac{F^{N-1}(s_j) - F^{N-1}(s_{t-1})}{N(F(s_j) - F(s_{t-1}))} \right) \]
\[= \ (s_i - b_{t+1})k_1 + (s_j - b_j)k_2 \]

\(^{11}\) Yu (1999) page 26 Lemma 6
where \( k_1, k_2 > 0 \) represent the probability of winning.

Recall from the definition of \( s_i \), we have that \( b_i < s_i < b_{i+1} \). Therefore, from equation (4.10) and Lemma 3, we have the following two conditions

1) As bidder's value approaches \( b_i \), \( k_1 \) will increase to fulfill equation (4.9)

2) As bidder's value approaches \( b_{i+1} \), \( k_2 \) will decrease to fulfill equation (4.9)

These two conditions imply the trade-off between the payoff and the probability of winning. As bidder's value approaches \( b_i \), the bidder submits a lower bid to ensure a positive payoff but the corresponding probability of winning decreases. On the other hand, as bidder's value approaches \( b_{i+1} \), the bidder submits a higher bid in which may lead to a negative payoff but the corresponding probability of winning increases. Recall that in a sealed-bid second-price auction, they actual paid price equals to the second highest bid. Hence, bidding higher than one's personal to the good may still lead to a positive payoff.

We can also conclude that, unlike the continuous bidding in which the truthful bidding is the dominant strategy, there is no dominant strategy in discrete bidding. This is because for \( v_i \in (b_i, b_{i+1}) \), it is possible for a bidder to choose either \( b_i \) or \( b_{i+1} \) to get a positive payoff, shown in equation (4.10). This means both underbidding and overbidding are not weakly dominated. In real world, one can choose a strategy of overbidding or underbidding based on his or her estimation of probability. For a person who overestimates the probability of winning with value \( v_i \), he or she might prefer to bid over \( s_i \) in order to get a higher payoff. On the other
hand, a person who overestimates the probability of winning with value \( v_i \) might prefer to bid below \( s_i \) to avoid potential risk by giving up some amount of payoff.

**E. The Expected Revenue for Sellers**

In the previous section, we have discussed the expected payoff for bidders. The expected payoff is the most important thing that bidder cares and the payoff function is related to the probability density function, the conditional probability of winning and the number of bidders. In this section, we will discuss the most important thing for sellers, which is the expected revenue.

In a sealed-bid second-price auction, the revenue of the seller equals to the second highest bid. Therefore, the expected revenue for seller is determined by the second highest bid level and its corresponding probability. Let \( M \) be the probability mass function for each bidder to bid \( b_i \).

Note that the probability mass function \( M \) has no direct relationship with the probability density function \( F(v) \) in the last section. Let \( E\Pi \) denote the expected revenue of the seller, then we have

\[
E\Pi = b_0 \Pr(SHB=b_0) + b_1 \Pr(SHB=b_1) + \ldots + b_{M+1} \Pr(SHB=b_{M+1})
\]  

(5.1)

Same as the previous notations, SHB stands for the second highest bid.

Now we are going to discuss the conditional probability. Consider \( \Pr(SHB=b_i) \), there are two cases such that \( SHB=b_i \).
Case 1: there is only one bidder bids \( b \geq b_{i+1} \), one or more bidders bid \( b_i \). This means that the highest bid is unique and there exists at least one second highest bid.

Case 2: two or more bidders bid \( b_i \) and no bidder bids \( b \geq b_{i+1} \). This means that there is a tie in the highest bid, in other word, the highest bid equals to the second highest bid.

Case 1 describes the situation in which there are \( k \) bidders with valuation in range of \([b_i, b_{i+1})\), one bidder with valuation in range of \([b_{i+1}, b_M]\) and other \( N-k-1 \) bidders with valuation in range of \([b_0, b_i)\). For those \( k \) bidders, the probability is denoted as \( (M(b_{i+1}) - M(b_i))^k \); for \( N-k-1 \) bidders, the probability is denoted as \( M(b_{i+1})^{N-k-1} \), for that one bidder, the probability is denoted as \( 1 - M(b_{i+1}) \). Hence, the probability of case 1 turns out to be

\[
\text{Prob}(\text{case1}) = \sum_{k=1}^{N-1} \binom{N-1}{k} (M(b_{i+1}) - M(b_i))^k M(b_{i+1})^{N-k-1} (1 - M(b_{i+1}))
\]

Case 2 describes the situation in which there \( k \) bidders with valuation in range of \([b_i, b_{i+1})\) and other \( N-k \) bidders with valuation in range of \([b_0, b_i)\). For those \( k \) bidders, the probability is denoted as \( (M(b_{i+1}) - M(b_i))^k \); for \( N-k \) bidders, the probability is denoted as \( M(b_i)^{N-k} \). Hence, the probability in case 2 is

\[
\text{Prob}(\text{case2}) = \sum_{k=1}^{N-1} \binom{N-1}{k} (M(b_{i+1}) - M(b_i))^k M(b_{i+1})^{N-k}
\]

Therefore, we can update equation (5.1) by plugging these probabilities as
\[
E\Pi = \sum_{i=0}^{M+1} b_i \left( \text{Prob(case1)} + \text{Prob(case2)} \right)
\]

\[
= \sum_{i=0}^{M+1} b_i \left[ \sum_{k=1}^{N-1} \binom{N-1}{k} (M(b_{i+1}) - M(b_i))^k M(b_{i+1})^{N-k-1} (1 - M(b_{i+1})) \right] 
+ \sum_{i=0}^{M+1} b_i \left[ \sum_{k=2}^{N} \binom{N}{k} (M(b_{i+1}) - M(b_i))^k M(b_{i+1})^{N-k} \right] 
\]

Equation (5.2) is the updated version of equation (5.1) having the probability mass function.

In order to increase the second highest bid, which is equivalent to seller's revenue, the seller could make the increment of bidding levels, \( M \), to be smaller so that the second highest bid may increase relative to the original one. As \( M \) approaches 0, the equilibrium of discrete bidding will converge to continuous bidding in which the truthful bidding dominates. Therefore, on average, seller would like to provide an incentive for bidder to bid on their true value for the good instead of underbidding. Hence, by decreasing the bidding incremental will certainly help seller to rise the expected revenue since bidders are more likely to bid on their true value for the good.

Another important factor that could affect the expected revenue for sellers is the number of bidders. In this case,

\[
\frac{\partial}{\partial N} E\Pi > 0
\]

Hence, it is better for seller to call for more bidders to attend the auction.
Chapter 5

Conclusion and Application

In my thesis, I studied the sealed-bid second-price auction with both continuous bidding and discrete bidding. In the case where the bidding space is continuous, truthful bidding is the dominant strategy for bidders. Bidders prefer to bid on their true value for the good to maximize their expected payoff and at the same time, sellers could maximize their expected revenue since all bidders will not underbid. In the case where the bidding space is discrete, we discussed the expected payoff for bidders and the expected revenue for sellers. Truthful bidding is no longer the dominant strategy and bidders will consider the trade-off between expected payoff and winning probability. On seller's side, the incremental size and the number of bidders affect the expected revenue. Based on these results, we can make changes to those factors to increase the expected revenue for sellers.

We have shown that there exists a trade-off between the expected payoff and the corresponding probability of winning. Bidding lower grants the bidder a lower chance of winning but ensure a non-negative expected payoff. Bidding higher enhances the winning probability for bidder along with a risk of having negative payoff. The bidders, knowing their probability density function, could calculate the equilibrium point $s_i$ and compare it to their own value for the good, $v_i$. By knowing the relationship between $s_i$ and $v_i$, a bidder could
maximize his or her expected payoff. In terms of the seller, the seller would always want the bidder to overbid.

For the expected revenue for sellers, we have discussed that there are two factors affecting it, namely the incremental size and the number of bidders. As the incremental size approaches to zero, the expected revenue approaches to that of in continuous bidding. The number of bidders has positive correlation with the expected revenue. With more bidders attending the auction, the sellers earns more expected revenue. In general, for any distribution of the bidders' valuations, the expected revenue in discrete bidding space is less than that in the continuous bidding space. Such lose in revenue is compensated in time saving and communication cost since the bidding procedure is shortened. Hence, if the cost of timing and communication are trivial, for example, online auction costs nearly no communication fee and timing for bid up, then it is better to shrink the incremental size to increase the expected revenue for sellers. As the bidding increment decreases, the expected revenue in discrete bidding would converge to that of in continuous bidding. Meanwhile, increasing the number of bidders would certainly compress the room for those who want to win the good by offering a relatively lower bid.


APPENDIX A

Proof of Proposition 1: Conditions b, c, d are the Lemma 1, 2, 3, respectively, in Yu's (1999) page 20.

For bidder \( i \), let \( E\pi(v_i, b_m) \) denote the expected payoff for bidder \( i \) with valuation \( v_i \) from bidding \( b_m \). WTS: for a bidder with \( v_i \leq b_j \), \( E\pi(v_i, b_j) \geq E\pi(v_i, b_m) \) for every \( b_m > b_j \);

for a bidder with \( v_i \geq b_j \), \( E\pi(v_i, b_j) \geq E\pi(v_i, b_m) \) for every \( b_m < b_j \).

Let \( p_j \) denote the probability for winner \( i \) bidding \( b_j \). The expected payoff function at this situation is

\[
E\pi(v_i, b_m) = \sum_{t=1}^{m-1} (v_i - b_t) p_t + \frac{1}{t} (v_i - b_m) p_m
\]

Consider \( E\pi(v_i, b_m) - E\pi(v_i, b_{m-k}) \)

\[
\frac{1}{2} ((v_i - b_m) p_m + (v_i - b_{m-k}) p_{m-k}) + \sum_{t=m-k-1}^{m-1} (v_i - b_t) p_t
\]

for \( k = 2, 3, ..., m \) and

\[
\frac{1}{2} ((v_i - b_m) p_m + (v_i - b_{m-1}) p_{m-1})
\]

for \( k = 1 \).

For bidder \( i \) with \( v_i \leq b_{m-k} \), equation (A) \( \leq 0 \). For bidder \( i \) with \( v_i \geq b_m \), (A) \( \geq 0 \).
Proof of (b): \( B_i = \{ v \in [0,1] \mid b(v) = b_i \} \) is convex in \( v \).\(^{12}\)

Suppose that \( b(v_1) = b(v_2) = b_i \) where \( v_1, v_2 \in [0,1] \), from the definition of expected payoff function, we have

\[
\begin{cases}
E\pi(v_1, b_i) \geq E\pi(v_j, b_i), \forall b_j \\
E\pi(v_1, b_i) \geq 0
\end{cases}
\]

and

\[
\begin{cases}
E\pi(v_2, b_i) \geq E\pi(v_j, b_i), \forall b_j \\
E\pi(v_2, b_i) \geq 0
\end{cases}
\]

Recall the definition of \( E\pi \), we have that

\[
v_i p_i - p_i p_j \geq v_j p_j - p_j p_j, \forall j
\]

\[
v_2 p_i - p_i p_j \geq v_2 p_j - p_j p_j, \forall j
\]

We conclude that

\[
v p_i - p_i p_j \geq v p_j - p_j p_j, \forall j
\]

with \( E\pi(v, b_i) \geq 0 \). \( \blacksquare \)

Proof of (c):

Suppose \( b(v) = 0 \) for all \( v \), then \( E\pi(v, b(v)) = \frac{1}{N} \) for all \( v \). If some bidder with

\[
v = \frac{1}{N} + \epsilon (\epsilon > 0)
\]

deviates and bids \( \frac{\epsilon}{2} \), he will win the good with probability of 1. We have

\(^{12}\) Yu (1999) page 55
Proof of (d):

From (4.1), we have that

\[ vP_i - p_i P_i \geq vP_j - p_j P_j, \forall j \neq i \]

Or

\[ v(P_i - P_j) \geq p_i P_j - p_j P_j, \forall j \neq i \]

Hence, for any \( v' > v \), we have that \( b(v') \geq b(v) \)

Proof of Lemma 3:

\[
E \pi \left( \frac{1 + \varepsilon}{N}, \frac{\varepsilon}{2} \right) \geq \frac{1}{N} + \varepsilon - \frac{\varepsilon}{2} \\
= \frac{1}{N} + \frac{\varepsilon}{2} \\
> \frac{1}{N} \\
= E \pi \left( \frac{1}{N}, \varepsilon, 0 \right)
\]

Proof of Lemma 3:

\[
\frac{F^N(s_i) - F^N(s_{i-1})}{N(F(s_i) - F(s_{i-1}))} = \frac{F^{N-1}(s_i) + F^{N-2}(s_i)F(s_{i-1}) + \ldots + F(s_i)F^{N-2}(s_{i-1}) + F^{N-1}(s_{i-1})}{N}
\]

Since \( s_{i-1} < s_i \), \( F(s_{i-1}) < F(s_i) \). Then \( F^{N-2}(s_i)F(s_{i-1}) < F^{N-1}(s_i) \),...

\[
F(s_i)F^{N-2}(s_{i-1}) < F^{N-1}(s_{i-1}), \quad F^{N-1}(s_{i-1}) < F^{N-1}(s_i) \]. This implies that

\[
F^{N-1}(s_i) + F^{N-2}(s_i)F(s_{i-1}) + \ldots + F(s_i)F^{N-2}(s_{i-1}) + F^{N-1}(s_{i-1}) < NF^{N-1}(s_i)
\]
and we have

\[
\frac{F^N(s_i) - F^N(s_{i-1})}{N(F(s_i) - F(s_{i-1}))} < F^{N-1}(s_i)
\]

Similarly, we have

\[
F^{N-1}(s_i) < \frac{F^N(s_{i+1}) - F^N(s_i)}{N(F(s_{i+1}) - F(s_i))}
\]

By combining the results, we have

\[
\frac{F^N(s_i) - F^N(s_{i-1})}{N(F(s_i) - F(s_{i-1}))} < F^{N-1}(s_i) < \frac{F^N(s_{i+1}) - F^N(s_i)}{N(F(s_{i+1}) - F(s_i))}
\]