The Module Approach to Tensors

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Abstract

This undergraduate thesis presentation poster offers an introduction to module theory with the purpose of helping bring an advanced undergraduate student of mathematics to a level of understanding of modules and of tensor products. Since these topics are each dense and abstract in their own right, we constructed this thesis work to expound both naturally. This presentation was inspired by various resources written on each topic of interest and was made with the purpose of exploring the tensor product and some of its implications through the lens of introductory module theory. We have constructed as concise and as clear as possible of an undergraduate level introduction to the topics of module theory and tensor products.

Basic Definitions in Module Theory

*Let R be a ring (not necessarily commutative nor with unity). A **left R-module** is a set M together with two things:*

Definition 1.1.

1. *A binary operation $+$ on M under which M is an abelian group; and*
2. *An action of R on M, i.e. a map $R \times M \to M$, $(r, m) \mapsto rm$ for all $r \in R$ and for all $m \in M$ which satisfies the following:*
   - *(a) $(r + s)m = rm + sm$ for all $r, s \in R$ and $m \in M$,*
   - *(b) $(rs)m = r(sm)$ for all $r, s \in R$ and $m \in M$,*
   - *(c) $r(m + n) = rm + rn$ for all $r \in R$, $m, n \in M$.*

   If R is a ring with unity, we additionally require:
   - *(d) $1m = m$ for all $m \in M$.*

Definition 1.2. *Let R be a ring and let M be an R-module. An **R-submodule** of M is a subgroup of M which is closed under the action of ring elements, i.e., $rn \in N$ for all $r \in R, n \in N$.*

Proposition 1.3. ***(The Submodule Criterion)*** *Let R be a ring and let M be an R-module. A subset N of M is a submodule of M if and only if*

1. *$N \neq \emptyset$, and*
2. *$x + ry \in N$ for all $r \in R$ and for all $x, y \in N$.*

Module Homomorphisms

Definition 2.4. *Let R be a ring and let M and N be R-modules.*

1. *A map $\varphi : M \to N$ is an **R-module homomorphism** if it respects the R-module structures of M and N:*
   - *(a) $\varphi(x + y) = \varphi(x) + \varphi(y)$ for all $x, y \in M$ and*
   - *(b) $\varphi(rx) = r\varphi(x)$ for all $r \in R$, $x \in M$.*
2. *An R-module homomorphism is an **isomorphism** (of R-modules) if it is both injective and surjective. The modules M and N are said to be isomorphic, denoted $M \cong N$, if there is some R-isomorphism $\varphi : M \to N$.*
3. *Let M and N be R-modules and define $\mathrm{Hom}_R(M, N)$ to be **the set of all R-module homomorphisms from M into N**.*
4. *If $\varphi : M \to N$ is an R-module homomorphism, let $\ker\varphi = \{m \in M \mid \varphi(m) = 0\}$ and let $\varphi(M) = \{n \in N \mid n = \varphi(M)$ for some $m \in M\}$. These are the **kernel** and **image** of φ, respectively.*

An Important Result of Module Homorphisms

Lemma 2.5. *Let M and N be R-modules of a ring R and $\varphi : M \to N$ an R-module homomorphism. Then $\ker\varphi$ is a submodule of M.*

Proof.

Let M and N be R-modules of a ring R and $\varphi : M \to N$ an R-module homomorphism. We claim that $\ker\varphi$ is a submodule of M. First, we know that since M is a module, it is an abelian group under addition. Thus it contains the additive identity element, 0. Since N is also an abelian group under addition and since φ is an R-module homomorphism, we require that $\varphi(0_M) = 0_N$. Thus $0_M \in \ker\varphi$ and hence $\ker\varphi \neq \emptyset$. Next, let $r \in R$, $x, y \in \ker\varphi$ be arbitrary. Then $x + ry \in \ker\varphi$ if and only if $\varphi(x + ry) = 0$. As $r \in R$ and $y \in M$, an R-module, $ry \in M$. Since φ is an R-module homomorphism, we have $\varphi(x + ry) = \varphi(x) + \varphi(ry)$. Now, $\varphi(x) = 0_N$ and $\varphi(ry) = r\varphi(y) = r * 0_N = 0_N$ as both $x, y \in \ker\varphi$. Thus $\varphi(x + ry) = 0_N$ and so $x + ry \in \ker\varphi$. Therefore $\ker\varphi$ is a submodule of M by the Submodule Criterion. □

The First Isomorphism Theorem

Theorem 2.6. ***(The First Isomorphism Theorem)*** *Let M, N be R-modules and let $\varphi : M \to N$ be an R-module homomorphism. Then $M/\ker\varphi \cong \varphi(M)$.*

Proof.

Let M and N be R-modules and let $\varphi : M \to N$ be an R-module homomorphism. For simplicity in this proof, we will use homomorphism and isomorphism to refer to R-module homomorphisms and R-module isomorphisms. Let $\mu : M/\ker\varphi \to \varphi(M)$ be defined by $\mu(x + \ker\varphi) = \varphi(x)$ for $x \in M$. To prove that this is in fact well defined, assume $x + \ker\varphi = y + \ker\varphi$ for some $x, y \in M$ with $x = y + z$ for some $z \in \ker\varphi$. Thus $\mu(x + \ker\varphi) = \varphi(x) = \varphi(y + z) = \varphi(y)\varphi(z) = \varphi(y)\mu(z + \ker\varphi) = \varphi(y) = \mu(y + \ker\varphi)$. Now we must show that μ is an isomorphism. In its construction, it may be obvious that it is a homomorphism. If it is not so obvious, let $x + \ker\varphi, y + \ker\varphi \in M/\ker\varphi$ and $r \in R$ be arbitrary. Thus as φ is a homomorphism:

$$\mu((x+\ker\varphi)+(y+\ker\varphi)) = \mu((x+y)\ker\varphi) = \varphi(x+y) = \varphi(x)+\varphi(y) = \mu(x+\ker\varphi)+\mu(y+\ker\varphi).$$

Similarly, $\mu(rx + \ker\varphi) = \varphi(rx) = r\varphi(x) = r\mu(x + \ker\varphi)$. Thus μ is a homomorphism, and we must now show that it is bijective. Let $y \in \varphi(M)$. Thus $y = \varphi(x)$ for some $x \in M$. But then $x + \ker\varphi \in M/\ker\varphi$ and so $\mu(x + \ker\varphi) = \varphi(x) = y$ as desired. Thus μ is surjective. Now let $a + \ker\varphi, b + \ker\varphi \in M/\ker\varphi$ be arbitrary with $\mu(a + \ker\varphi) = \mu(b + \ker\varphi)$. Thus $\varphi(a) = \varphi(b)$. Since φ is a homomorphism, we have that $\varphi(a - b) = 0$. Thus $a - b \in \ker\varphi$. So then $a - b = z$ for some $z \in \ker\varphi$. Thus $a = z + b$ and so $a + \ker\varphi = (z + b) + \ker\varphi = b + \ker\varphi$ and μ is injective and hence an isomorphism as desired. □

The Theorem of Universal Property

Theorem 3.7. ***(The Theorem of Universal Property)*** *For any set A there is a free R-module $F(A)$ on the set A and $F(A)$ satisfies the following universal property:*

If M is any R-module and $\varphi : A \to M$ is any map of sets, then there is a unique R-module homomorphism $\Phi : F(A) \to M$ such that $\Phi(a) = \phi(a)$, for all $a \in A$.

$$\begin{tikzcd} A \arrow[r, "\text{Inclusion}"] \arrow[dr, "\varphi"'] & F(A) \arrow[d, "\Phi"] \\ & M \end{tikzcd}$$

When A is the finite set [illegible], $F(A) \cong R \oplus R \oplus \cdots \oplus R \cong R^n$

The Tensor Product of Modules

Definition 4.8. *Let, first, R and S be two rings where R is a subring of S. Then define N as an R-module and take S as its own S-module. We require that N be an abelian group. Additionally, we invoke a map $S \times N \to N$ defined by $(s, n) \mapsto sn$. We then consider the free $\mathbb{Z}$-module on the set $S \times N$. This is the free abelian group. This free abelian group on the set $S \times N$ is the collection of all finite commuting sums of elements of the form (s_i, n_i) with $s_i \in S$ and $n_i \in N$. To satisfy the necessary terms of an S-module, define T as the quotient of N and the subgroup H generated by all elements of the form*

$$(s_1 + s_2, n) - (s_1, n) - (s_2, n),$$
$$(s, n_1 + n_2) - (s, n_1) - (s, n_2), \text{ and}$$
$$(sr, n) - (s, rn)$$

*for $s_1, s_2, s \in S$, $n_1, n_2, n \in N$, and $r \in R$. It is this quotient group that we call the **tensor product of S and N over R**, denoted $S \otimes_R N$.*

Definition 4.9. *The elements of the previously defined quotient group are called **tensors**. Each may be expressed as finite sums of **simple tensors** of the form $s \otimes n$ with $s \in S$ and $n \in N$.*

The Universal Property of the Tensor Product

Theorem 4.10. *(**The Universal Property of the Tensor Product**) Suppose R is a ring with unity, M is a right R-module, and N is a left R-module. Let $\iota : M \times N \to M \otimes_R N$ be the R-balanced map defined by $(m, n) \mapsto m \otimes n$. Then the following is true:*

1. *If $\Phi : M \otimes_R N \to L$ is any group homomorphism from $M \otimes_R N$ to an abelian group L then the composite map $\varphi = \Phi \circ \iota$ is an R-balanced map from $M \times N$ to L.*
2. *Conversely, suppose L is an abelian group and $\varphi : M \times N \to L$ is any R-balanced map. Then there is a unique group homomorphism $\Phi : M \otimes_R N \to L$ such that φ factors through ι, i.e., $\varphi = \Phi \circ \iota$.*

This diagram demonstrates this property visually as before:

$$\begin{tikzcd} M \times N \arrow[r, "\iota"] \arrow[dr, "\varphi"'] & M \otimes_R N \arrow[d, "\Phi"] \\ & L \end{tikzcd}$$

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