INCENTIVE TO INNOVATE: DYNAMIC OPTIMIZATION STRATEGY IN THE CASE OF A SYMMETRIC DUOPOLY

by

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ABSTRACT

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Technological dominance and spillovers play important roles in a firm's decision to investment in innovated products. It is intuitive to think that a firm which is technologically ahead will dominate the market for innovated products. However, the question of the spillover advantage a firm gets when they are technologically behind makes the decision to invest in new technology more complex. Therefore, in this paper, I consider the investment in new product and cost of doing research, along with capital and level of technology, to be primary factors affecting a firm's profit. I ask, when is it a good time to invest in new product and when is it appropriate for a firm to allocate more funding for research? I find that firms tend to do more research when they have more market share and invest less when the total capital in the market increases. They tend to invest more when they are technologically ahead and do more research when they are behind. This is a dynamic game because a firm's decision to invest depends not only on its own level of technology but also on the rival firm's level of technology and market share. The presence of technological spillovers also adds onto the dynamics of the game since it discourages the firm, which is technologically ahead, from investing in technological advances.

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CHAPTER ONE

INTRODUCTION

In this project, I am asking, what is the optimal dynamic[1] strategy to maximize profit when firms decide to innovate? I am primarily interested in investigating this in the context of symmetric duopoly where there are two main firms selling identical products and there are many buyers in the market. One good example of firms in such market structure is Airbus and Boeing. I use their product timeline to get a sense of the pattern that my model has to fit. We can observe that Boeing and Airbus introduces new aircraft model in response to the rival firm. My structural model tries to replicate the patterns that are similar to their behaviors. It is common to see Boeing and Airbus introducing new products in the market. Since this is a duopoly, firms compete for the market share of their products. So then, what are the factors that affect the investment decisions? What happens when the firms collude and/or compete?

Technological dominance and spillovers also play important roles in the investment decision of a firm. It is intuitive to think that a firm which is technologically ahead will dominate the market for innovated products, like in the case of Boeing and Airbus. However, the question of spillover advantage that a firm gets when they are technologically behind makes the decision to invest in new technology more complex. Thus, in this paper I consider the investment in new productive capacity and cost of doing research, along with capital and level of technology, to be primary factors affecting a firm's profit. In their paper, Dawid et al. (2010) present their model in the form of two firms producing homogeneous goods with one of the firms having the option to innovate (i.e. to introduce a new product) which is differentiated vertically and horizontally from

[1] Dynamic optimization refers to the process of minimizing or maximizing some objective function over a period of time.

the previous products. Building onto their paper, I test if the outcome differs when we give both firms the option to innovate.

I am mainly interested in identifying the optimal level of investment in new productive capacity and the optimal level of research that determines the optimal level of capital and technology that will maximize profit. This is a dynamic game because a firm's decision to invest depends not only on its level own level of technology but also on the rival firm's level of technology and market share. The presence of technological spillovers also adds onto the dynamics of the game since it discourages the firm that is technologically ahead from investing in the new products.

I find that firms tend to do more research when they have more market share and invest less when the total capital in the market increases. They tend to invest more when they are technologically ahead and do more research when they are behind. This is most probably because when the rival firm is dominating the market and has superior technology, a firm would want to increase research to catch up. In doing so, they can also reduce the cost of doing research since they can take advantage of reduced research cost due to technological spillover. This enables the firm that is technologically behind to invest in production when they are trying to catch up technologically as well.

My paper is organized as follow. Chapter 2 provides insight to the pattern of innovation of Boeing and Airbus and the background information on how Boeing and Airbus came to be a duopoly. The second part of chapter 2 introduces the mathematical tools (*Calculus of Variations* and *Optimal Control Theory*) that I use for solving the equations. Full excerpts from the sources of these topics are provided in the appendices as well. This chapter also provides a brief history on the evolution of dynamic optimization theory. Finally, in the last part, I provide an example of dynamic optimization problem and a walk through of the process involved in solving the problem using optimal control.

After that, chapter 3 discusses the literatures on various topics and researches in the field of dynamic optimization in a duopoly setting. In addition, it also covers papers on the effect of R&D spillovers and product innovation. Then, first part of chapter 4 covers the setting up the model and the objective functions for the firms. Then the later part provides conjectures and proofs based on the optimal solution paths of the state and control variables. Finally, chapter 5 concludes the paper.

CHAPTER TWO

BACKGROUND

2.1. Market Structure

Past few decades has been a period of constant innovation and competition between Boeing and Airbus. Both firms consistently introduced new models in the market and made new modifications to the old ones in order to stay ahead of the game. Table 1 shows the list of all the family of commercial aircraft that Boeing and Airbus introduced so far.

Boeing Co.		**Airbus**	
Year Introduced	Aircraft Model	Year Introduced	Aircraft Model
1968	737	1972	A300
1970	747	1983	A310
1982	767	1988	A320
1983	757	1993	A330
1995	777	1994	A340
2011	787	2007	A380

Table 1. *Year of commercial aircraft models introduced by Boeing and Airbus. Data collected from the company websites: www.boeing.com and www.airbus.com .*

We can observe that the timing of most of the models introduced were close to the rival firm's timing. By illustrating the information in a timeline format, it is easy to see the pattern in Boeing and Airbus' behavior in introducing new aircraft models. Figure 1 illustrates the information in Table 1 as a timeline. There is a pattern in the timing of innovation by both firms: whenever one firm innovates, the other one does so within a year or two, and then there are no new innovations for a long time. For instance, Airbus entered the market in 1972 with A300 in order to challenge and compete with Boeing's 747 (1970). Then the next technological breakthrough does not occur until 1982 when Boeing introduced the 767. A year later, in 1983, Airbus introduced A310. After that, there is no innovation for next five years and then Airbus introduced A320 in 1988, A330 in 1993 and A340 in 1994. To follow up, Boeing introduced 777 in 1995. Once again, there was no innovation for a long time after that. Finally, in 2007, Airbus introduced A380 in the market followed up by Boeing with 787 (Dreamliner) in 2011.

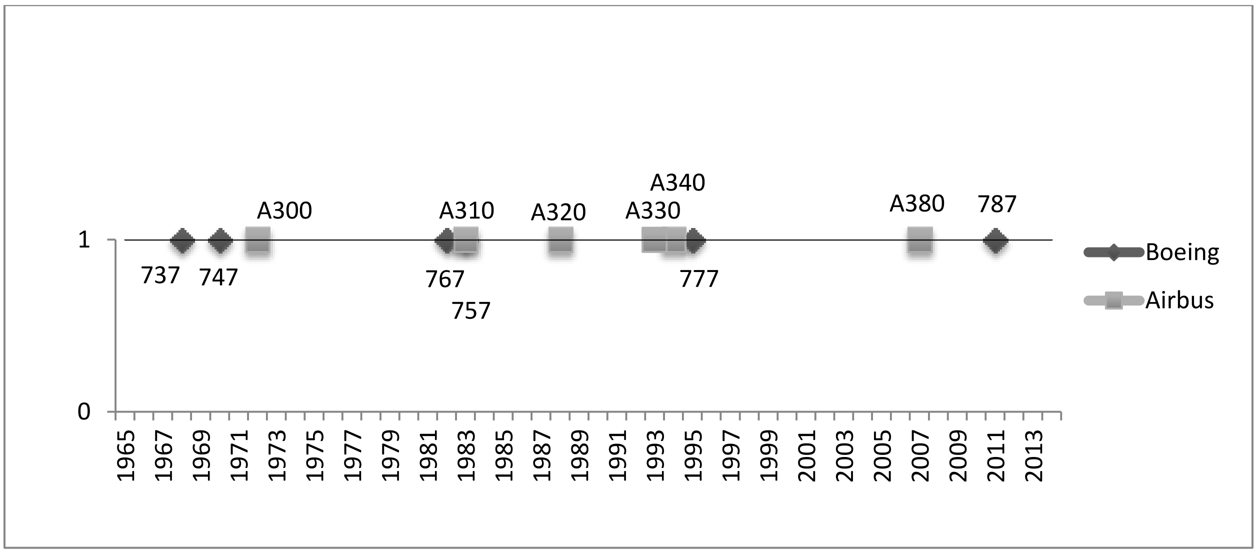

Figure 1. *Timeline of commercial aircraft models introduced by Boeing and Airbus. Data collected from the company websites: www.boeing.com and www.airbus.com*

Boeing and Airbus are in duopoly market structure. So we need some model of duopoly to reflect the competition. However, the model evolves over time so we need a dynamic model. First we will explore how these firms came to be a duopoly. Since late 1960s, there have been only two big names in the commercial aircraft industry: Boeing and Airbus. Series of events resulted in a duopoly in the global market between Airbus and Boeing.

Named after its founder, William Boeing, Boeing Co. (BA) is one of the oldest aircraft manufacturing company which dates back to 1916.[2] It has seen the rise and fall of many partner and rival firms alike and matured as a company over the years. For instance, Douglas Aircraft Co. (another successful aircraft manufacturer) was one of Boeing's biggest rivals in the 1960s. Douglas Aircraft was best known for "DC" (Douglas Commercial) commercial aircraft series. However after the end of World War II, the company was struggling to expand its production and eventually went through a merger with McDonnell Aircraft Corp. (another key producer of military aircraft) to form the McDonnell Douglas Corp.[3] McDonnell Douglas struggled to maintain the profitability of its commercial aircraft division (Olienyk et al., 2011) which then on August 1, 1997, lead to merger with Boeing to form the present day Boeing Company. [4] As for Lockheed (another big rival firm), the company was unable to realize profitability in the commercial aircraft production and had to withdraw from the business in the early 1980s. Thus this marked the end of an era of competition on the home ground making Boeing a monopoly in the U.S. Since then, Boeing has managed to dominate the aircraft market and create barriers to entry (especially in the U.S. market).

However, the competition from the outside was just on the rise with the agreement of French, German and British governments on plans to build European aircraft.

"...A joint statement states the governments have agreed "for the purpose of strengthening European co-operation in the field of aviation technology and thereby promoting economic and

[2] Details available on company website: www.boeing.com
[3]The Boeing Logbook: 1964 – 1970, http://www.boeing.com/boeing/history/chronology/chron10.page? . retrieved on 11/11/2013.
[4] [4]The Boeing Logbook: 1997 – 2001, http://www.boeing.com/boeing/history/chronology/chron16.page#97; retrieved on 11/11/2013.

technological progress in Europe, to take appropriate measures for the joint development and production of an Airbus". The aim is also to challenge American domination."[5]

Airbus was founded as a consortium in December, 1970, when Airbus Industry was formally established as a *Groupement d'Interet Économique* (Economic Interest Group or GIE). This comprised of France's Aerospatiale (a merger of Sud Aviation, SEREB and Nord Aviation) with 50% ownership and Germany's Deutsche Airbus (a group of four firms, Messerschmittwerke, Hamburger Flugzeugbau, VFW GmbH and Siebelwerke ATG) with the other 50% stake. At the time when Airbus was established, the global market for commercial aircraft was dominated by American firms such as Boeing, Douglas Aircraft and Lockheed with 90% market share. With the commitment from British, German, French and Spanish governments to provide financial support (in the form of loans with low interest rates) Airbus was able to survive and expand its market share in the commercial aircraft industry.

2.2. Dynamic Optimization Theory

Airbus and Boeing are playing a dynamic game (evolving over time) against one another, so we need to understand how to solve dynamic optimization problems to explain their behavior. Following are some of the terminologies and definitions that will help understand the background on the type of game and mathematical tools that are used in this paper. First, let us look at the mathematical tools that are frequently used in economics to solve a dynamic optimization problem. Then we will look into a brief history of evolution of dynamic optimization theory in economics.

[5] The Timeline: July 1967, http://www.airbus.com/company/history/the-timeline/ retrieved on 11/11/2013

2.2.1. Mathematical Tools

We look at the dynamic optimization techniques of the calculus of variations and of optimal control theory as analytic methods for solving planning problems in continuous time. Likewise, these tools are used by economists to solve problems involving duopoly models like mine.

a) Calculus of Variation[6]

The origin of calculus of variations can be traced back to the posing of the brachistochrone problem[7] by John Bernoulli in 1696 and its solution by him and independently by his brother James in 1697. Other specific problems were solved and a general mathematical theory was developed by Euler and Lagrange.

The modern era began in the early 1960s with a resurgence of interest by mathematicians and groups of economists and management scientists in certain dynamic problems. The optimal control theory was developed in Russia by Pontryagin and his co-workers in the late 1950s. It was then translated into English in 1962. The theory generalizes the calculus of variations by extending its range of applicability. Functional optimization (Appendix.1) using *Gateaux Variation* allows us to derive the Euler-Lagrange equation. We are then able to use the equation to solve calculus of variation problems.

b) Optimal Control

The *maximum principle* for optimal control was developed in the late 1950's by L. S. Pontryagin and his co-workers. The maximum principle applies to all calculus of variations problems as optimal control gives equivalent results. However, the two approaches differ and

[6] The definitions and history are derived from Kamien and Schwartz (1991).
[7] The brachistrochrone problem posed by John Bernoulli was, "*Given two points A and B in a vertical plane, what is the curve traced out by a point acted on only by gravity, which starts at A and reaches B in the shortest time.*"

sometimes optimal control approach gives insights into a problem that might be less readily apparent through the calculus of variations.

Optimal control (see Appendix. 2) also applies to problems for which the calculus of variations is not convenient, especially the ones that involve constraints on the derivatives of functions sought. In optimal control problems, variables are divided into two classes, *state* variables and *control* variables. The movement of state variables is governed by first order differential equations.

2.2.2. Brief History[8]

Frank Ramsey's (1928) paper on the optimum rate of savings of a nation is largely known as the paper which introduces the application of dynamic optimization in economic problems. He analyzed a continuous-time dynamic optimization model, and developed a modification of the standard calculus of variations method to show the existence of an optimum savings rate, when all generations over an infinite time horizon are to be treated equally in the objective function (Majumdar et al., 2000). Over the last forty years, there has been fast growth in the field of industrial economics involving both theoretical and empirical (George et al, 1992). Economists and management scientists alike have attempted to generalize the theory of firm and find simplified applications in the field of industrial economics. Almost after thirty years since Ramsey, the method of optimal control devised by Pontryagin et al. (1962) lead to a renewed interest in the field of dynamic optimization. Number of papers studied Ramsey's problem in depth including Cass (1965), Srinivasan (1964) and Koopmans (1965).

[8] excerpts from Majumdar et al. (2000), Chapter 2 of the book

The 1950's and early 60's gave rise to a literature on finite-horizon pure capital accumulation oriented dynamic optimization exercises, where optimality was defined in terms of only the state of the economy at the end of the horizon. The dynamic optimization problems of the Ramsey-type were viewed differently after the shift in emphasis of many economies away from planning at the national level. The problem being solved was now viewed as a descriptive problem that a typical representative agent solves rather than a normative problem the "social planner" ought to solve. The Ramsey objection to discounting future utilities as "ethically indefensible" on the part of the social planner was no longer relevant. Thus, the problem to be solved in describing an agent's behavior would now be a discounted dynamic optimization problem of the Ramsey-type. This reformulation of the subject had two important outcomes: the issue of the existence of an optimal program, which had occupied center-stage for undiscounted dynamic optimization models, became a relatively unimportant aspect of the theory for discounted model; and description of dynamic behavior of optimal programs became considerably more difficult.

In this project, we are mainly concerned about the oligopolistic market structure and the behaviors of firms in such market. Every dynamic optimization problems have a time horizon. It can be discrete or continuous or infinite. For many problems in economic interest, future values are discounted. In infinite horizon problems, a transversality condition needed to provide a boundary condition is usually replaced by the assumption that the optimal solution approaches a steady state. It is a reasonable assumption since in the long run the optimal solution would tend to stabilize as the environment is stationary by hypothesis.[9] Let us look at an example of a dynamic optimization problem in economics and the method used to solve it.

Example 2. [10]Let *P(x)* be the profit rate that can be earned with a stock of productive capital *x*, where *P'*(0) > 0 and *P''* < 0. The capital stock decays at a constant proportionate rate $b \geq 0$. Investment cost is an increasing convex function of the gross investment rate *u*, with $C'(0) = 0$ and *C''* > 0. We seek the investment rate *u(t)* that maximizes the present value of the profit stream over infinite time horizon.

$$\max \int_0^{\infty} e^{-rt}[P(x) - C(u)]\, dt$$
$$\text{subject to} \qquad x' = u - bx, \;\; x(0) = x_0 > 0, \qquad u \geq 0.$$

We use the *current value Hamiltonian* (Appendix 2) to characterize the solution. Since it is typically convenient for us to analyze and discuss in terms of current values, rather than present value (i.e. values at time *t* rather than their equivalent at time zero). In addition, the differential equations describing an optimum solution will be autonomous when the multiplier is given in its current value form if the state and the control variables do not depend on *t* explicitly (see Appendix 2).

The current value Hamiltonian is

$$H = P(x) - C(u) + m(u - bx)$$

If the optimal investment rate is positive, it satisfies

$$C'(u) = m$$

[9] Page 174, Kamien and Schwartz (1991)
[10] Page 140 and 166, Kamien and Schwartz (1991).

where the current value multiplier[11] obeys

$$m' = (r + b)m - P'(x)$$

The solution *x*, *u*, *m* must satisfy the forgoing conditions (if it involves u > 0). We cannot find the solution explicitly without specification of *P* and *C*. Nevertheless, we can qualitatively characterize the solution by sketching paths compatible with the conditions for either the *x*-*m* plane or the *x*-*u* plane. For the detailed solution on this example problem, refer to page 166 of Kamien and Schwartz (1991). This technique is useful for analyzing dynamic behavior of firms and we shall explore some of the papers that make use of such technique to solve dynamic problems in the next chapter.

[11] The current value multiplier, *m*(t), gives the marginal value of the state variable (i.e. *P*(*x*)) at time *t* in terms of values at *t*.

CHAPTER THREE

LITERATURE REVIEW

This topic is related to the literature on accumulation games, where capacity investments of single product firms engaged in oligopolistic competition have been characterized both in the framework of open-loop Nash equilibria and Markov-perfect Nash equilibria (Dawid et al., 2010). This section also examines some of the existing literature on innovation investment and as well as on dynamic competition in duopoly. It also includes literatures on games with (and without) product differentiation, symmetric and asymmetric market structure and finally on competition and cooperation between firms.

As defined by Dockner et al. (1999), a capital accumulation game follows as a result of the case where capital serves as a strategic variable in production decisions. Dockner et al. (1999) put together a set of dynamic investment games consisting of capital accumulation games and study existence and qualitative properties of Markov Perfect Equilibria (MPE). A Markov Perfect Equilibrium is a pair of reaction functions that form a perfect equilibrium. A Markov strategy is a strategy that only depends on the payoff-relevant state variables and a state transition equation of some kind that affects the profit function (Maskin et al., 1987). In their paper, Dockner et al. consider a firm that invests in a capital stock. They formulate a discrete-time dynamic game that fits into the class of capital accumulation games. There are two firms, each investing in a capital stock with linear investment costs. Like my paper, their paper considers a constant depreciation rate of capital stock and on the capital stock of the rival firm. They are able to characterize two different types of MPE in their result. One is a degenerate but strict equilibrium that results in constant investment over time and the other one is the indifferent MPE which is not strict and has the property that for a given equilibrium strategy of one player, every response of the opponent is the best response. Similarly, Fershtman and Muller (1983) look into capital accumulation games of infinite duration by considering a market in which firms accumulate capital according to Nerlove-Arrow capital accumulation equation. They formulate an open loop solution[12] and provide a proof for the existence to the infinite horizon case of open-loop, nonzero sum, differential games. They are also able to show the convergence to a stationary equilibrium regardless of the initial stocks of capital.

In relation to investment in innovation and capital accumulation game, we need to consider the inventory capacity and productive capacity for the new products. For instance, Berg et al. (2012) studies the case of duopoly with intertemporal capacity constraints where producers have limited number of products for two consecutive periods. They do this in the context of dynamic Cournot duopoly[13] with and without commitment and find that flexible supply contracts can adversely affect welfare when production precedes sales and firms face an intertemporal capacity constraint. They find that there is a unique pure strategy Nash equilibrium for any allocation of initial supplies under commitment, whereas under non-commitment it is possible that a subgame perfect Nash equilibrium in pure strategies may not exist. With commitment, prices slowly increase over time and an increase in stocks (assuming all else being equal) results in high profits. In contrast, they find that, prices may decline and increasing stocks may lead to lower profits absent commitment. Although larger firms typically prefer not to commit, commitment is beneficial for smaller firms, society at large and for buyers.

Like my paper, a number of papers explore topics in R&D spillovers and incentives to innovate. For example, Cellini and Lambertini (2009) study the dynamic R&D for innovation in

[12] Open loop solution is a solution in which the strategy of the opponent does not affect the decision (strategy) of a player as he cannot observe the strategy used by his opponent

[13] In a Cournot competition, firms compete on the quantity of output they produce. Their decisions to produce are simultaneous and independent from each other.

a Cournot duopoly under two different settings: competition and cooperation. They consider a duopoly with homogeneous goods over continuous time. Likewise, I also set up my model in a continuous time horizon even though the goods need not necessarily be homogeneous. In their paper, Cellini et al. adopt an explicitly dynamic approach to describe the R&D activity meant for process innovation, modeled as a differential game. They limit themselves to the cases where firms either play a non-corporative game or form a R&D cartel. They then compare the steady state profits and social welfare at the subgame perfect equilibria of each case. They find that regardless of the level of spillover, firms prefer R&D cooperation to non-cooperative behavior from both private and social point of view. Their analysis shows that a unique stable (in the sense of saddle point) equilibrium exists in each case. The analysis of the model shows that higher the level of technological spillover, the larger the present value of investment efforts over time, under both non-cooperative and cartel. Similarly, Narajabad and Watson (2011) try to include the dynamics of horizontal differentiation (Hotelling competition of product location) in the context of innovation and competition. They compute Markov-perfect equilibria and analyze the effects of changes in transportation costs and product relocation cost on long-run innovation. They find that innovation tends to increase when firms are located near each other than when they are separated.

Also, Femminis and Martini (2010) study the incentive to be first mover in the game of innovation. Their intuition is that, in an oligopoly, the first mover can possibly have a competitive advantage such as higher quality products and lower cost of production. However, being ahead of everyone entails high R&D costs and the risk of takeovers by competitor in subsequent improvements. Furthermore, they also point out that there are chances that the new technology may fail to generate profits or provide undesired knowledge spillovers to rival firms.

My paper is also motivated by similar intuitions. They consider a process innovation framework in which the R&D activity of the firm that is ahead generates a technological spillover. The new information is disclosed after a time period that name “disclosure lag”. In the presence of disclosure lags, being ahead has two offsetting effects on profitability: the leader enjoys temporary competitive advantage (for the duration of the disclosure lag) but also pays higher R&D costs. The technologically inferior firm makes lower profits while catching up to the leader’s technological level, but benefits from new information through the spillover. They identify a new type of equilibrium in the presence of technological spillovers and of a disclosure lag. The pioneer firm delays investment until the R&D cost is low enough that the follower will find it optimal to invest as soon as he benefits from the spillover (i.e. immediately after the disclosure lag).

Femminis and Martini (2011) extend on their previous paper and examine the advantage of being a second (or later) mover when there are R&D spillovers. Their model belongs to the category of symmetric stopping time game, which can be sub-categorized depending on the presence of a first-mover or a second-mover advantage. They show that spillovers substantially affect the equilibrium of dynamic game. They identify an equilibrium in which the firms’ behavior is different from that of the previous papers. Even for low level spillovers, the leader delays investment until the stochastic fundamental is high enough that the follower finds it optimal to invest as soon as he obtains the spillover. In this equilibrium, the existence of the spillover determines the length of the leader’s cost-advantage period. It is in the leader’s best interest to wait until the market condition makes their investment worthwhile the limited cost-advantage period. To follow suit, it is in the best interest of the follower to invest as soon as he benefits from the information spillover because his fixed costs are lower (owing to the spillover).

The behavior of the follower depends on the information available about the new technology. If the follower has access to the relevant information; the follower finds it optimal to wait and delay his investment when the demand is low; while he finds it optimal to invest right away when the demand is high. When there is no spillover, the follower finds it optimal to invest and pay the full cost only when demand has reached high values. On the other hand, the follower waits and tries to benefit from the spillover when the demand is low.

However, existing literatures have largely neglected the importance of optimal innovation strategies in the context of oligopolistic market. Dawid et al. (2010) are few of the people who have dealt with capacity adjustment processes when firms produce more than one product. They test how the option to invest in the new market affects capacity dynamics of both firms. They also ask, what is the value of innovation option for the innovating firm and how is the non-innovating incumbent affected by the innovation (option) of its rival? They find that the non-innovating firm benefits most of the time.

My paper differs from the previous ones in the sense that I have incorporated both technological level and investment in productive capacity as a factor for a firm's profit. Technology and productive capacity is further affected by the level of R&D investment and investment in new innovated products. This indicates the existence of subgame perfect equilibrium since firms choose the best investment strategy in response to their rival firm's choice of investment and consequently the technological level of the rival firm. Unlike Dawid et al., I give both of the firms the option to invest in the new innovated product (i.e. the technological advance) which may or may not be differentiated. Thus, this leads to a symmetric game which yields different outcome as compared to their asymmetric game.

CHAPTER FOUR

MODEL AND ANALYSIS

My paper is concerned about the incentive to innovate when there is a technological spillover. My objective is to model a duopoly market structure in which a firms decision to invest in innovation is affected by the presence of technological spillovers. Thus, I plan to present a model that has technological spillovers that tends to keep firms together in technological progress.

Since I am modeling a duopoly market structure, I assume that there are only two firms in the market. I let K_i be the level of capital of firm $i \in \{1,2\}$. The initial production capacity is $K_i(0) = K_i^{ini} \geq 0$. T_i is the technology level of firm i. The initial level of technology is $T_i(0) = T_i^{ini} \geq 0$. Assuming the similar assumptions from Dawid et al., both firms exploit their production capacities and price of firm *i* is then given by the following linear demand system:

$$p_i(t) = 1 - \left(K_i(t) + K_j(t)\right) + \varphi\left((T_i(t) - T_j(t)\right) \quad (1)$$

where, $0 < \varphi \leq 1$ represents the degree of the effect of net technological level on price. To simplify the problem, there are no marginal production costs and so the production costs are linear. The total cost of the firm is represented by the quadratic function of investment and research costs, i.e.

$$C_i\left(I_i(t),\, R_i(t)\right) = \left[\frac{c_i I_i(t)^2}{2}\right] + \left[\frac{d_i R_i(t)^2}{2}\right] \quad (2)$$

where $I_i(t)$ denotes the investment in capital and $R_i(t)$ denotes the research of firm *i* = 1, 2 at time *t*, and c and *d* are positive parameters. Now there are four relevant state variables (capacities) which evolve according to the state dynamic:

$$K'_i(t) = I_i - \delta K_i(t), \tag{3}$$

$$T'_i(t) = R_i(t) - \phi\left(T_i(t) - T_j(t)\right) - \gamma T_i(t), \tag{4}$$

Where $i \in \{1,2\}$, and $\phi\left(T_i(t) - T_j(t)\right)$ represents the R&D spillover. δ is the depreciation rate of the capital stock and so if the firm does not put effort into maintaining the capital it depreciates over time. Similarly, γ is the depreciation rate of technology which implies that if firms do not put some effort in maintaining the level of technology, it depreciates over time.

The initial conditions are

$$K_i(0) = K_i^{ini} \geq 0, \quad T_i(0) = T_i^{ini} \geq 0, \tag{5}$$

which states that, at time 0 the value of capital and technology is positive.

The instantaneous profit function of firm *i* is then given by

$$\pi_i = \left\{1 - \left(K_i + K_j\right) + \varphi(T_i - T_j)\right\} K_i - \left(\frac{c_i}{2} I_i^2\right) - \left(\frac{d_i}{2} R_i^2\right) \tag{6}$$

Now we can define the marginal profit of firm *i* with respect to capital

$$\frac{d\pi_i}{dK_i} = \varphi(T_i - T_j) + 1 - 2K_i - K_j \tag{7}$$

We can see that the marginal profit of firm *i* is positive but decreasing, which implies that the marginal value of more capital diminishes as the firm adds more capital. Moreover, the marginal profit also depends on technology which shows that the firm with higher technology benefits more from investing in production.

Firm 1 and 2 both chooses their investments in order to maximize their discounted profit net of investment and research costs over an infinite time horizon. The discount rate is denoted by $r > 0$. The objective function for firm i is then given by

$$J_i = \int_0^{\infty} e^{-rt}\left[\{1-(K_i+K_j)+\ \varphi(T_i-T_j)\}K_i - \left(\frac{c_i}{2}I_i^2\right)-\left(\frac{d_i}{2}R_i^2\right)\right]dt \tag{8}$$

Subject to:

$$K'_i = I_i - \delta K_i, \quad K_i(0) = K_i^{ini} \geq 0, \tag{9}$$

$$T'_i = R_i - \phi(T_i - T_j) - \gamma T_i, \quad T_i(0) = T_i^{ini} \geq 0, \tag{10}$$

Where $i \in \{1, 2\}$ and $j \in \{2, 1\}$

The Hamiltonian for firm i is

$$H(I,R,\lambda,t) = e^{-rt}\left[\{1-(K_i+K_j)+\ \varphi(T_i-T_j)\}K_i-\left(\frac{c_i}{2}I_i^2\right)-\left(\frac{d_i}{2}R_i^2\right)\right]+\lambda_i(I_i-\delta K_i) + \lambda_{Ti}(R_i-\phi(T_i-T_i)-\gamma T_i) \tag{11}$$

where $\lambda_i$is the shadow value (marginal value) of K_i and λ_{Ti} is the shadow value of T_i.

***Theorem 1**: The optimal Investment and Research at steady state (i.e. $I_i' = 0$ and $R_i' = 0$) is*

$$I_i^* = \frac{1-2K_i-K_j+\varphi(T_i-T_j)}{(r+\delta)c_i} \text{ and } R_i^* = \frac{\varphi K_i}{(r+\phi+\gamma)d_i}$$

Proof:

The current value Hamiltonian for firm i is

$$\mathcal{H} \equiv e^{rt}H = \left[\{1-(K_i+K_j)+\ \varphi(T_i-T_j)\}K_i-\left(\frac{c_i}{2}I_i^2\right)-\left(\frac{d_i}{2}R_i^2\right)\right]+m_i(I_i-\delta K_i) + m_{Ti}(R_i-\phi(T_i-T_j)-\gamma T_i) \tag{12}$$

where we define $m_i \equiv e^{rt}\lambda_i$ and $m_{Ti} \equiv e^{rt}\lambda_{Ti}$ as the current value multiplier. m_i is the shadow value of K_i evaluated at time *t* and m_{Ti} is the shadow value of T_i at time *t*.

The relevant first order conditions (FOCs) for the optimum are:

$$\frac{\partial\mathcal{H}}{\partial I_i} = 0 \quad \Rightarrow \quad -c_iI_i + m_i = 0 \tag{13}$$

$$\frac{\partial\mathcal{H}}{\partial R_i} = 0 \quad \Rightarrow \quad -d_iR_i + m_{Ti} = 0 \tag{14}$$

$$\frac{\partial \mathcal{H}}{\partial\, m_i} = K'_i \qquad \Rightarrow \qquad I_i - \delta K_i = K'_i \tag{15}$$

$$\frac{\partial \mathcal{H}}{\partial m_{Ti}} = T'_i \qquad \Rightarrow \qquad R_i - \phi\left(T_i - T_j\right) - \gamma T_i = T'_i \tag{16}$$

Equation (13) states that, the marginal cost of investment must equal the marginal value m_i of a unit of additional capital. Likewise, equation (14) states that, the marginal cost of doing research must equal the marginal value m_{Ti} of a unit of additional technology.

From (13) and (14) we get

$$I_i = \frac{m_i}{c_i} \tag{17}$$

$$R_i = \frac{m_{Ti}}{d_i} \tag{18}$$

On the basis of the above FOCs on control variables, the co-state equations for the solutions are

$$rm_i - \frac{\partial \mathcal{H}}{\partial K_i} = m'_i \qquad \Rightarrow \qquad (r+\delta)m_i - 1 + 2K_i + K_j - \varphi(T_i - T_j) = m'_i \tag{19}$$

$$rm_{Ti} - \frac{\partial \mathcal{H}}{\partial T_i} = m'_{Ti} \qquad \Rightarrow \qquad (r + \phi + \gamma)m_{Ti} - \varphi K_i = m'_{Ti} \tag{20}$$

These conditions must be evaluated along with the initial conditions

$$K_i(0) = K_i^{ini} \geq 0, \quad T_i(0) = T_i^{ini} \geq 0$$

We differentiate (17) and (18) totally and use them to eliminate m_i and m_{Ti} and use (19) and (20) to eliminate m'_i and m'_{Ti} from the results. Thus the resulting system is equation (15), (16) and

$$\begin{aligned} I'_i = \frac{m'_i}{c_i} &= \frac{(r+\delta)m_i - 1 + 2K_i + K_j - \varphi(T_i - T_j)}{c_i} \\ &= \frac{(r+\delta)c_i I_i - 1 + 2K_i + K_j - \varphi(T_i - T_j)}{c_i} \end{aligned} \tag{21}$$

$$R_i' = \frac{m_{Ti}'}{d_i} = \frac{(r + \phi + \gamma)m_{Ti} - \varphi K_i}{d_i}$$
$$= \frac{(r + \phi + \gamma)d_i R_i - \varphi K_i}{d_i} \qquad (22)$$

From equation (21), the points (K_i, I_i, T_i) such that $I_i' = 0$ satisfy

$$I_i^* = \frac{1 - \left(2K_i + K_j\right) + \varphi(T_i - T_j)}{(r + \delta)c_i} \qquad (23)$$

And the equation (22) with the points (K_i, R_i, T_i) such that $R_i' = 0$ satisfy

$$R_i^* = \frac{\varphi K_i}{(r+\phi+\gamma)d_i} \qquad (24)$$

Since this problem is infinite horizon and autonomous (see Appendix 2, part c), the steady state is defined by $m'_i = m_{Ti}' = K'_i = T'_i = 0$.

The optimal level of investment for each firm is dependent on twice the amount of one's capital and the capital of the rival firm and the net technology level of the firms. Thus, suggesting that the solution for optimal investment is closed-loop subgame Nash equilibrium (i.e. Nash equilibrium for the subgame in which firm *i*'s strategies depend not only on time but also on current values of observed state variables of the rival firm). At steady state, the optimal investment decreases for any further increase in capital of both firms. This suggests that, there is a decreasing return to investment in capital. So, beyond the point of optimal investment, it is not profitable to invest further in productive capacity as it lowers instantaneous profit of the firm. On the other hand, being technologically ahead indicates that the firm should invest more. This is most probably because of the fact that firms can take advantage of reduced production cost (as found by Femminis and Martini (2010)). Another possible explanation is that, as technology rises, the price also rises and hence makes it more attractive for firms to invest more in productive capacity.

The research level of firm *i* is only dependent on its own capital. This suggests that the solution is open-loop Nash equilibrium (i.e. each firm's decision depends only on time) for the optimal level of research. In open-loop equilibria, each player ignores the strategy played by the other firm. The optimal solution shows that firms increase research when they have higher level of capital. Since both firms have the incentive to be ahead of the other firm, it makes intuitive sense that they have the incentive to do more research when they are technologically ahead and has higher level of capital. Furthermore, since capital depreciates over time, K'_i will not tend to infinity as $t \to \infty$ (if δ is sufficiently high enough) and thus the level of research will also be finitely bounded.

In general, for the effect of technology on investment we can consider the following cases:

Case 1: $\boldsymbol{T_i > T_j}$

Lemma 1: at I_i^*, if $T_i > T_j$, *then firm i's investment increases as the technological gap* (ΔT) *increases.*

Proof:

Assume that $T_i > T_j$. Then the term $\varphi(T_i - T_j)$ in equation (23) is positive

So let $|T_i - T_j| = \Delta T$, then

$$I_i^* = \frac{1 - (2K_i + K_j) + \varphi(\Delta T)}{(r + \delta)c_i} \tag{25}$$

Thus the result implies that when firm *i*'s technology is greater than the technology of its rival firm, investment of firm *i* is positively correlated to the net technological level (ΔT). Hence, all else equal, increase in T_i increases investment in this case.

Case 2: $T_i < T_j$

***Lemma 2*:** at I_i^* , if $T_i < T_j$*, then firm i's investment decreases with increase in technological gap.*

Proof:

Assume that $T_i < T_j$. Then the rival firm is technologically ahead which means that the term $\varphi(T_i - T_j)$ in equation (23) is negative

So let $T_i - T_j = \Delta T$, where $\Delta T < 0$. Then

$$I_i^* = \frac{1 - \left(2K_i + K_j\right) - \varphi(\Delta T)}{(r + \delta)c_i} \qquad (26)$$

Equation (26) implies that the technological gap is negatively correlated to investment decision in this case. Thus, the investment of firm *i* decrease as the technological gap rises.

Case 3: $T_i = T_j$

Lemma 3: *If* $T_i - T_j = 0$ *(i.e. the firms are at same level of technology) then the optimal investment path is not affected by the level of technology.*

Proof :

Assume that $T_i - T_j = 0$, then from equation (23) we have

$$I_i^* = \frac{1 - (2K_i + K_j) - \varphi(T_i - T_j)}{(r + \delta)c_i}$$

$$I_i^* = \frac{1 - (2K_i + K_j)}{(r + \delta)c_i} \quad (27)$$

At $T_i - T_j = 0$ the effect of technology on the investment decision vanishes.

Theorem 2*: When firm i is technologically behind, they will do more research and vice versa. Thus in the long run firms will tend to converge towards* $T_i = T_j$

Proof:

Using Lemma 1, 2, and 3 and the assumption that firms have the option to use their resources to either invest and/or do research, we can conclude that, in a duopoly, firms have the tendency to try and catch up to the technological level of their rivals. Given the symmetric nature of the game, we can then conclude that the firms tend to converge towards $T_i = T_j$ (since both firms have the incentive to move towards $T_i = T_j$ according to Lemma 1 and 2). Hence, all else being equal, I_i^* achieves maximum at $T_i = T_j$ with respect to the optimal level of technology.

Proposition 1: *Optimal technological level is a function of own research and the technological level of the rival firm. That is, firm i's technology increases as the firm's research increases and as well as when the rival firm's technology increases. Furthermore, at optimal research level, state of technology is positively correlated to the capital.*

Proof:

From (11), setting $\mathrm{T}'_i = 0$, we have

$$R_i - \phi\left(T_i - T_j\right) - \gamma T_i = 0$$

So, $$T_i^* = \frac{1}{\phi+\gamma} R_i + \frac{\phi}{\phi+\gamma} T_j \quad (28)$$

This implies that technology is an increasing function of a firm's research and rival firm's level of technology. In other words, firms play the catch up game and so, if the rival firm increases his technological level then I (the corresponding firm *i*) need to catch up to him as well. This does indeed replicate the behaviors of Boeing and Airbus. Both Boeing and Airbus tend to innovate simultaneously: if one innovates the other follows suit soon after.

At $R_i = R_i^*$, substituting (24) into (28), we then get

$$T_i{}^* = \frac{\varphi K_i}{(r+\phi+\gamma)d_i(\phi+\gamma)} + \frac{\phi T_j}{\phi+\gamma} \qquad (29)$$

which means that, at optimal level of research, higher level of capital results in higher level of technology.

From (15), setting $K'_i = 0$, we have

$$I_i - \delta K_i = 0$$

$$\text{so,} \quad K_i{}^* = \frac{I_i}{\delta} \qquad (30)$$

At $I_i = I_i^*$, substituting (18) into (24), we then get the steady state path for K

$$K_i{}^* = \frac{1 - \left(2K_i^* + K_j\right) + \varphi(T_i - T_j)}{(r+\delta)\delta c_i}$$

$$K_i{}^* = \frac{1 - K_j + \varphi(T_i - T_j)}{(r+\delta)\delta c_i + 2} \qquad (31)$$

Theorem 3: *The optimal level of capital and technology for firm i is*

$$K_i^* = \frac{1}{\theta}\left(1 + \frac{\varphi}{\theta\beta - \varphi} - K_j\left(1 + \frac{\varphi}{\theta\beta - \varphi}\right) - \varphi T_j\left(\left(\frac{\varphi}{\theta\beta} - \frac{\phi}{\phi + \gamma}\right)\left(\frac{\theta\beta}{\theta\beta - \varphi}\right) - 1\right)\right)$$

and

$$T_i^* = \frac{1 - K_j}{\theta\beta - \varphi} - T_j\left(\frac{\varphi}{\theta\beta} - \frac{\phi}{\phi + \gamma}\right)\left(\frac{\theta\beta}{\theta\beta - \varphi}\right)$$

where $\theta = (r + \delta)\delta c_i + 2$ and $\beta = (r + \phi + \gamma)d_i(\phi + \gamma)$.

Proof:

From (29) we can solve for K_i and get

$$K_i = (r + \phi + \gamma)d_i(\phi + \gamma)\left(T_i^* - \frac{\phi T_j}{\phi + \gamma}\right) \tag{32}$$

Evaluating at $K_i = K_i^*$ and $T_i = T_i^*$ we can set (31) equal to (32)

$$\frac{1 - K_j + \varphi(T_i^* - T_j)}{(r + \delta)\delta c_i + 2} = (r + \phi + \gamma)d_i(\phi + \gamma)\left(T_i^* - \frac{\phi T_j}{\phi + \gamma}\right)$$

For ease of calculation, let $\theta = (r + \delta)\delta c_i + 2$ and $\beta = (r + \phi + \gamma)d_i(\phi + \gamma)$

Then

$$\frac{1 - K_j + \varphi T_i^* - \varphi T_j}{\theta} = \beta\left(T_i^* - \frac{\phi}{\phi + \gamma} T_j\right)$$

$$\frac{1 - K_j + \varphi T_i^* - \varphi T_j}{\theta\beta} = T_i^* - \frac{\phi}{\phi + \gamma} T_j$$

$$T_i^* - \frac{\varphi}{\theta\beta}T_i^* = \frac{1 - K_j}{\theta\beta} - \frac{\varphi}{\theta\beta}T_j + \frac{\phi}{\phi + \gamma} T_j$$

$$T_i{}^* \left(\frac{\theta\beta - \varphi}{\theta\beta}\right) = \frac{1 - K_j}{\theta\beta} - T_j \left(\frac{\varphi}{\theta\beta} - \frac{\phi}{\phi + \gamma}\right)$$

$$T_i{}^* = \frac{\frac{1 - K_j}{\theta\beta} - T_j \left(\frac{\varphi}{\theta\beta} - \frac{\phi}{\phi + \gamma}\right)}{\left(\frac{\theta\beta - \varphi}{\theta\beta}\right)}$$

$$T_i{}^* = \frac{1 - K_j}{\theta\beta - \varphi} - T_j \left(\frac{\varphi}{\theta\beta} - \frac{\phi}{\phi + \gamma}\right)\left(\frac{\theta\beta}{\theta\beta - \varphi}\right) \qquad (33)$$

Then substituting (33) into (31) we get

$$K_i{}^* = \frac{1 - K_j + \varphi \left(\frac{1 - K_j}{\theta\beta - \varphi} - T_j \left(\frac{\varphi}{\theta\beta} - \frac{\phi}{\phi + \gamma}\right)\left(\frac{\theta\beta}{\theta\beta - \varphi}\right) - T_j\right)}{\theta}$$

$$K_i{}^* = \frac{1 - K_j + \varphi \left(\frac{1 - K_j}{\theta\beta - \varphi}\right) - \varphi T_j \left(\left(\frac{\varphi}{\theta\beta} - \frac{\phi}{\phi + \gamma}\right)\left(\frac{\theta\beta}{\theta\beta - \varphi}\right) - 1\right)}{\theta}$$

$$K_i{}^* = \frac{1}{\theta}\left(1 + \frac{\varphi}{\theta\beta - \varphi} - K_j \left(1 + \frac{\varphi}{\theta\beta - \varphi}\right) - \varphi T_j \left(\left(\frac{\varphi}{\theta\beta} - \frac{\phi}{\phi + \gamma}\right)\left(\frac{\theta\beta}{\theta\beta - \varphi}\right) - 1\right)\right) \qquad (34)$$

Equation (33) states that technological level of the firm *i* is subjected to decrease as the rival firm's capital and technological level rises. Similarly equation (34) states that the market share of firm *i* is subjected to decreases as the rival firm gains more market share and dominates the market with higher level of technology. Since this is a symmetric duopoly market the steady state rule applies to both firms.

CHAPTER FIVE
CONCLUSION

From the analysis in previous chapter, I find that firms tend to do more research when they have more capital and invest less when the total capital in the market increases. At steady state, the optimal investment decreases for any further increase in capital of both firms. This suggests that, there is a decreasing return to investment in capital. So, beyond the point of optimal investment, it is not profitable to invest further in productive capacity.

They tend to invest more when they are technologically ahead and do more research when they are behind. This is most probably because firms can take advantage of reduced production cost (as found by Femminis and Martini (2010)). Moreover, when the rival firm is dominating the market and has superior technology, a firm would want to increase research to catch up. In doing so, they can also reduce the cost of doing research since they can take advantage of reduced research cost due to technological spillover. This enables the firm that is technologically behind to invest in production when they are trying to catch up technologically as well. Firms also do more research when they have higher level of capital. Since both firms have the incentive to be ahead of the other firm, it makes intuitive sense that they have the incentive to do more research when they are technologically ahead.

I find that my model is able to reflect the behaviors of Boeing and Airbus. As mentioned in chapter 2, Boeing and Airbus consistently introduced new models of aircraft in the market in order to stay ahead in the game. As shown in table 1, the timing of most of the models introduced was close to the rival firm's timing. We can observe that the timing of the innovation was usually in response to the rival firm's innovation. This is reflected by the fact that both firms try to catch up to the other firm when they fall behind technologically. In the historical data, we can also see that both Boeing and Airbus introduced new models within ten years after the introduction of the previous one. In other words, there are no innovations in aircrafts for a long time, but when one firm innovates, the other one follows shortly after and then the pattern repeats. The long gap from one innovation period to the next one could possibly be explained by using *Theorem 2* in previous chapter. As I showed that since both firms have the tendency to catch up to the technological level of rival firm, when they achieve that equality in technological level, they have the incentive to at least stay at the same level of technology as the rival firm. Furthermore, whenever there is a new innovation in the market we showed that firm that is behind has the advantage of reduced research cost due to spillovers and thus, is able to catch up sooner than it would have taken if he was to innovate on his own. In that sense, the long gap could also represent the time taken for the firm that is ahead to research and achieve the technological breakthrough (i.e. new innovation).

There is lot of scope for future research in this field. In my paper, I was not able to provide complete analysis using phase diagrams due to limited time. One can possible use dynamic programming to do a graphical analysis of the model. It would also be interesting to find out how firms behave when the outcome of the investment in new products is uncertain. This will add the stochastic attribute to the model and it can possibly represent the reality more accurately. Nevertheless, given all the limitations, my model does have a close correlation to that of the market that I wanted to replicate, that is, Boeing and Airbus.

APPENDIX 1

Functional Optimization

$$\max_x \qquad J(x) = \int_{t_0}^{t_1} F\big(t, x(t), x'(t)\big)dt \tag{1.1}$$

$$\text{subject to} \quad x(t_0) = x_0\,, \quad x(t_1) = x_1$$

Gateaux Variation (First Variation) of J in the direction h is:

$$\delta\, \mathrm{J}\,(x; h) = \frac{d}{da} J(x + ah)\Big|_{a=0} \tag{1.2}$$

$$J(x + ah) = \int_{t_0}^{t_1} F(t, (x + ah), (x + ah)')dt \tag{1.3}$$

$$\frac{d}{da} J(x + ah) = \int_{t_0}^{t_1} \frac{d}{da} F(t, (x + ah), (x + ah)')dt$$

$$\frac{d}{da} F(t, (x + ah), (x + ah)')$$

$$= \frac{\partial F}{\partial x}(t, (x + ah), (x + ah)')h(t) + \frac{\partial F}{\partial x'}(t, (x + ah), (x + ah)')h'(t)$$

$$\frac{d}{da} J(x + ah)\Big|_{a=0} = \int_{t_0}^{t_1} \frac{\partial F}{\partial x} h(t) + \frac{\partial F}{\partial x'} h'(t)dt \tag{1.4}$$

$$\int_{t_0}^{t_1} \frac{\partial F}{\partial x'} h'(t)dt$$ [14]

Integration by parts:[15]

$u = \frac{\partial F}{\partial x'}\,; \; du = \frac{d}{dt}\left(\frac{\partial F}{\partial x'}\right)dt; \quad dv = h'(t); \quad v = h(t);$

$$\int_{t_0}^{t_1} \frac{\partial F}{\partial x'} h'(t)dt = \frac{\partial F}{\partial x} h(t)\Big|_{t_0}^{t_1} - \int_{t_0}^{t_1} h(t) \frac{d}{dt}\left(\frac{\partial F}{\partial x'}\right)dt$$

$$s\ ubject\ to \quad h(t_0) = 0\,, \quad h(t_1) = 0$$

$$\frac{d}{da} J(x + ah)\Big|_{a=0} = \int_{t_0}^{t_1} \frac{\partial F}{\partial x} h(t)dt - \int_{t_0}^{t_1} h(t) \frac{d}{dt}\left(\frac{\partial F}{\partial x'}\right)dt \tag{1.5}$$

$$= \int_{t_0}^{t_1} \left(\frac{\partial F}{\partial x} - \frac{d}{dt}\left(\frac{\partial F}{\partial x'}\right)\right) h(t)dt\ ; \qquad \forall\, h(t)\colon h(t_0) = 0\,, \quad h(t_1) = 0$$

Therefore,

$$\frac{\partial F}{\partial x} - \frac{d}{dt}\left(\frac{\partial F}{\partial x'}\right) = 0 \tag{1.6}$$ [16]

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[14] We need to get rid of $h'(t)$

[15] $\int u\, dv = uv - \int v\, du$

[16] Euler-Lagrange equation

APPENDIX 2

Optimal Control:

a. Simplest control problem

The simplest control problem is one of selecting a piecewise continuous control function $u(t), t_0 \leq t \leq t_1$, to

$$max \int_{t_0}^{t_1} f\big(t, x(t), u(t)\big)dt \qquad (2.1)$$

$$subject\ to \quad x'(t) = g\big(t, x(t), u(t)\big), \qquad (2.2)$$

$$t_0, t_1, \quad x(t_0) = x_0\ \ fixed;\ \ x(t_1) free. \qquad (2.3)$$

Here *f* and g are assumed to be continuously differentiable functions of these independent arguments, none of which is a derivatives. The *control* variable *u(t)* must be a piecewise continuous function of time. The *state* variable *x(t)* changes over time according to the differential equation (2.14) governing its own movement. The control *u* affects the objective (2.13), both directly and indirectly.

The calculus of variations problem of choosing a continuously differentiable function $x(t), t_0 \leq t \leq t_1$, to

$$max \int_{t_0}^{t_1} f\big(t, x(t), x'(t)\big)dt$$

$$subject\ to \quad x(t_0) = x_0$$

is easily transformed into an equivalent problem in optimal control. Let $u(t) = x'(t)$. Then the equivalent optimal control problem is

$$max \int_{t_0}^{t_1} f\big(t, x(t), u(t)\big) dt$$

$$subject\ to \quad x'(t) = u(t), \qquad x(t_0) = x_0$$

The state variable is *x*, while *u* is the control.

b. Simplest Problem – Necessary Conditions **(Part II, section 2 [Kamien, 91])**

To find necessary conditions that a maximizing solution $u^*(t), x^*(t), t_0 \leq t \leq t_1$, we follow a procedure similar to solving a nonlinear programming problem with Lagrange multipliers. Since the constraining relation (2.2) must hold at each *t* over the entire interval $t_0 \leq t \leq t_1$, we have a multiplier function $\lambda(t)$. Let $\lambda(t)$ be any continuously differentiable function *t* on $t_0 \leq t \leq t_1$.

For any function *x*, *u* satisfying (2.2) and (2.3) and any continuously differentiable function λ, all defined on $t_0 \leq t \leq t_1$, we have

$$\int_{t_0}^{t_1} f\big(t, x(t), u(t)\big) dt$$

$$= \int_{t_0}^{t_1} \big[f\big(t, x(t), u(t)\big) + \lambda(t) g\big(t, x(t), u(t)\big) - \lambda(t) x'(t)\big] dt \qquad (2.4)$$

since the coefficients of $\lambda(t)$ must sum to zero if (2.2) is satisfied, as we assume. Integrate the last term on the right of (2.4) by parts

$$\int_{t_0}^{t_1} -\lambda(t)x'(t)dt = -\lambda(t_1)x(t_1) + \lambda(t_0)x(t_0) + \int_{t_0}^{t_1} x(t)\lambda'(t)dt$$

Substitute into equation (2.4)

$$\int_{t_0}^{t_1} f\big(t, x(t), u(t)\big)dt =$$

$$\int_{t_0}^{t_1} \big[f\big(t, x(t), u(t)\big) + \lambda(t)g\big(t, x(t), u(t)\big) + x(t)\lambda'(t)\big]dt - \lambda(t_1)x(t_1) + \lambda(t_0)x(t_0) \qquad (2.5)$$

A control function *u(t)*, $t_0 \leq t \leq t_1$, together with the initial condition (2.3) and the differential equation (2.4) determine the path of the corresponding state variable *x*(t)*, $t_0 \leq t \leq t_1$. Since selection of the control function *u(t)* determines the state variable *x(t)*, it determines the value of (2.5) as well.

We consider a one-parameter family of comparison controls *u*(t)* + *ah(t)*, where *u*(t)* is the optimal control, *h(t)* is some fixed function, and *a* is a parameter. Let *y(t, a),* $t_0 \leq t \leq t_1$, denote the state variable generated by (2.2) and (2.3) with control *u*(t)* + *ah(t)*, $t_0 \leq t \leq t_1$. We assume that *y(t, a)* is a smooth function of both its arguments. So *a=0* provides the optimal path x^*.

$$y(t, 0) = x^*(t), \qquad y(t_0, a) = x_0$$

With the functions *u**, *x** and *h* all held fixed, the value of (2.1) evaluated along the control function *u*(t)* + *ah(t)* and the corresponding state *y(t, a)* depends on the single parameter *a*. We write

$$J(a) \quad = \int_{t_0}^{t_1} f(t, x(t), u^*(t) + ah(t))dt$$

Using (2.5),

$$J(a) \quad = \int_{t_0}^{t_1} [f(t, y(t,a), u^*(t) + ah(t)) + \lambda(t) g(t, y(t,a), u^*(t) + ah(t)) + y(t,a)\lambda'(t)]dt$$

$$- \lambda(t_1) y(t_1, a) + \lambda(t_0) y(t_0, a)$$

Since *u** is a maximizing control, the function *J(a)* assumes its maximum at *a=0*. Therefore, $J'(a) = 0$. Differentiating w.r.t. *a* and evaluating at *a=0* gives, on collecting terms,

$$J'(a) = \int_{t_0}^{t_1} [(f_x, \lambda g_x + \lambda') y_a + (f_u + \lambda g_u) h] dt - \lambda(t_1) y_a(t_1, 0) \qquad (2.6)$$

Now let λ obey the linear differential equation.

$$\lambda' = -[(f_x(t, x^*, u^*) + \lambda(t) g_x(t, x^*, u^*)], \qquad \text{with } \lambda(t_1) = 0$$

With λ given in the above equation, (2.6) holds provided that

$$\int_{t_0}^{t_1} [f_u(t, x^*, u^*) + \lambda g_u(t, x^*, u^*)] h dt = 0$$

for an arbitrary function *h(t)*. In particular, it must hold for $h(t) = f_u(t, x^*, u^*) + \lambda g_u(t, x^*, u^*)$, so that

$$\int_{t_0}^{t_1} [f_u(t, x^*(t), u^*(t)) + \lambda(t) g_u(t, x^*(t), u^*(t))]^2 dt = 0$$

This, in turn, implies the necessary condition that

$$f_u\big(t, x^*(t), u^*(t)\big) + \lambda(t) g_u\big(t, x^*(t), u^*(t)\big) = 0, \quad t_0 \le t \le t_1$$

So we have shown that if the functions *u*(t)*, *x*(t)* maximize (2.1), subject to (2.2) and (2.3), then there is a continuously differentiable function $\lambda(t)$ such that *u**,*x**, λ simultaneously satisfy the *state equation*

$$x'(t) = g\big(t, x(t), u(t)\big), \quad x(t_0) = x_0, \tag{2.7}$$

the multiplier equation

$$\lambda' = -\big[(f_x\big(t, x(t), u(t)\big) + \lambda(t) g_x\big(t, x(t), u(t)\big)\big], \qquad \lambda(t_1) = 0 \tag{2.8}$$

and the *optimality condition*

$$f_u(t, x(t), u(t)) + \lambda(t) g_u(t, x(t), u(t)) = 0 \tag{2.9}$$

for $t_0 \le t \le t_1$.

The way to remembering, or generating these conditions is the *Hamiltonian*

$$H\big(t, x(t), u(t), \lambda(t)\big) \equiv f(t, x, u) + \lambda g(t, x, u) \tag{2.10}$$

Now,

$$\frac{\partial H}{\partial u} = 0 \; genreates \; (2.9): \qquad \frac{\partial H}{\partial u} = f_u + \lambda g_u = 0$$

$$-\frac{\partial H}{\partial x} = \lambda' genreates \; (2.8): \qquad \lambda'(t) = -\frac{\partial H}{\partial x} = -(f_x + \lambda g_x),$$

$$\frac{\partial H}{\partial \lambda} = x' \; genreates \; (2.7): \qquad x' = \frac{\partial H}{\partial \lambda} = g$$

In addition, we have $x(t_0) = x_0$ and $\lambda(t_1) = 0$. At each *t*, *u* is a stationary point of the Hamiltonian for the given value of *x* and λ.

c. Discounting, Current Values, Comparative Dynamics **(Part II, section 8, [Kamien, 91])**

For many problems of economic interest, future values of rewards and expenditures are discounted, say, at the rate *r*:

$$max \int_0^T e^{-rt} f(t, x, u)dt \tag{2.11}$$

$$subject\ to \quad x'(t) = g(t, x, u), \qquad x(0) = x_0 \tag{2.12}$$

In terms of the Hamiltonian

$$H = e^{-rt} f(t, x, u) + \lambda g(t, x, u) \tag{2.13}$$

We require (x, u, λ) to satisfy,

$$H_u = e^{-rt} f_u + \lambda g_u = 0, \tag{2.14}$$

$$\lambda' = -H_x = e^{-rt} f_u - \lambda g_u = 0, \quad \lambda(T) = 0 \tag{2.15}$$

All values are discounted back to time 0; in particular, the multiplier $\lambda(t)$ gives a marginal valuation of the state variable at *t* discounted back to time zero. As illustrated below, it is often convenient to conduct the discussion in terms of current values. Further, if *t* is not an explicit argument of *f* and g, then the differential equations describing an optimum solution will be autonomous when the multiplier is given in its current form.

Write (2.13) in the form

$$H = e^{-rt}[f(t, x, u) + e^{rt} \lambda g(t, x, u)] \tag{2.16}$$

and define

$$m(t) \equiv e^{rt} \lambda(t) \tag{2.17}$$

as the *current value multiplier* associated with (2.12). Whereas $\lambda(t)$ gives the marginal value of the state variable at *t*, discounted back to time zero (when the whole problem is being solved), the new current value multiplier *m(t)* gives the marginal value of the state variable at time *t* in terms of values at *t*.

Also let

$$\mathcal{H} \equiv e^{rt}H = f(t,x,u) + mg(t,x,u) \tag{2.18}$$

We call $\mathcal{H}$ the *current value Hamiltonian.* Differentiating (2.17) with respect to time gives

$$\begin{aligned} m' &= re^{rt}\lambda + \lambda'^{e^{rt}} \\ &= rm - e^{rt}H_x \end{aligned} \tag{2.19}$$

on substituting from (2.17) and (2.15). In view of (2.18), $H = e^{-rt}\mathcal{H}$, so (2.19) becomes

$$\begin{aligned} m' &= rm - e^{rt}\frac{\partial(e^{-rt}\mathcal{H})}{\partial x} \\ &= rm - e^{rt}e^{-rt}\mathcal{H}_x \\ &= rm - f_x - mg_x \end{aligned} \tag{2.20}$$

In addition, (2.14) can be written as

$$H_u = \frac{\partial(e^{-rt}\mathcal{H})}{\partial u} = e^{-rt}\frac{\partial \mathcal{H}}{\partial u} = 0,$$

Which implies

$$\frac{\partial \mathcal{H}}{\partial u} = 0 \tag{2.21}$$

Finally, (2.12) may be recovered in terms of the current value Hamiltonian:

$$x' = \frac{\partial \mathcal{H}}{\partial m} = g \tag{2.22}$$

In sum, (2.13) – (2.15) may be equivalently stated as

$$\mathcal{H} = f(t,x,u) + mg(t,x,u) \tag{2.23}$$

$$\frac{\partial \mathcal{H}}{\partial u} = f_u + mg_u = 0, \tag{2.24}$$

$$m' = rm - \frac{\partial H}{\partial x} = rm - f_x - mg_x \tag{2.25}$$

Terminal conditions may be stated in terms of the current value multipliers by using conditions already derived and definition (2.17). For example, if $x(T)$ is free, then $\lambda(t) = e^{-rT}m(T) = 0$ is required. If $x(T) \geq 0$ is needed, then $e^{-rT}m(T) \geq 0$ and $e^{-rT}m(T)x(T) = 0$.

Notice first that (2.24) and (2.25) do not contain any discounted terms. Second, note that if t is not an explicit argument of f and g, then Equations (2.12), (2.24), and (2.25) reduce to

$$x' = g(x, u),$$

$$f_u(x, u) + mg_u(x, u) = 0,$$

$$m' = rm - f_x(x, u) - mg_x(x, u),$$

which is an *autonomous* (meaning time enters only through the discount term) set of equations; that is, they do not depend on time explicitly. Solving the second equation for $u = u(x, m)$ in terms of m and x and substituting into the equation for x', m' results in a pair of autonomous differential equations. In general, autonomous differential equations are easier to solve than non-autonomous ones. Furthermore, even if an explicit solution is not possible, phase diagram analysis of the qualitative properties of the solution may be possible when there are autonomous equations.

APPENDIX 3

Dawid et al. Model outline

Consider a duopoly market with Firm 1 and Firm 2, with initial production capacities $K_{ai}(0)(i = 1,2)$ available for an existing product (product a). Dawid et al. gives only Firm 1 the option to innovate and then Firm 2 is just a non-innovating incumbent. They assume that Firm 1 has just introduced an innovated new product (product b) and now it has to build up a production capacity for the new product. The initial production capacity of the product 2 is $K_b(0) = 0$.

They assume that both firms exploit their production capacities and prices are given by the following linear demand system:

$$p_a(t) = 1 - \left(K_{a1}(t) + K_{a2}(t)\right) - \eta K_{b1}(t),$$

$$p_b(t) = 1 - \eta\left(K_{a1}(t) + K_{a2}(t)\right) - \gamma K_{b1}(t).$$

For the parameter γ $(0 < \gamma \leq 1)$, if the $\gamma \leq 1$ holds, it indicates that the new product is vertically differentiated from the existing product. The parameter η represents the level of horizontal differentiation. Moreover, $\eta < \gamma$.

They also assume that there are no marginal production costs and so the production costs are linear. Investment costs are quadratic, that is, $C_i\left(I_{ij}(t)\right) = \frac{c_i I_{ij}(t)^2}{2}$, where $I_{ij}(t)$ denotes the investment of firm $j \in \{1,2\}$ for product *i* = a, b at time *t* and c_a and c_b are positive parameters.

Since Firm 1 is the only one innovating, there are three relevant state variables (capacities) which evolve according to the state dynamic:

$$\dot{K}_{a1} = I_{a1} - \delta_a K_{a1},$$

$$\dot{K}_{b1} = I_{b1} - \delta_b K_{b1},$$

$$\dot{K}_{a2} = I_{a2} - \delta_a K_{a2},$$

With initial conditions

$$K_{a1}(0) = K_{a1}^{ini} \geq 0, \quad K_{b1}(0) = 0, \qquad K_{a2}(0) = K_{a2}^{ini} \geq 0.$$

Firm 1 and 2 both chooses its investments in order to maximize its discounted profit net of investment costs over an infinite time horizon. The discount rate is denoted by $r > 0$. The objective functions are given by:

$$J_1 = \int_0^\infty e^{-rt} \left[(1 - (K_{a1} + K_{a2}) - \eta K_{b1}) K_{a1} + (1 - \eta (K_{a1} + K_{a2}) - \gamma K_{b1}) K_{b1} - \frac{c_a}{2} I_{a1}^2 - \frac{c_b}{2} I_{b1}^2 \right] dt$$

$$J_2 = \int_0^\infty e^{-rt} \left[(1 - (K_{a1} + K_{a2}) - \eta K_{b1}) K_{a2} - \frac{c_a}{2} I_{a2}^2 \right] dt$$

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