Cantor's Infinity

Michael Warrener

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1 Introduction

At the heart of mathematics is the quest to find patterns and order in some set of similar structures, whether these be shapes, functions, or even numbers themselves. In the late 1800's, there was a strong focus in the mathematical community on the study of real numbers and sequences of real numbers. Mathematicians quickly realized, however, that in order to do any meaningful investigation into the properties of sequences of real numbers, they needed a better definition of real numbers than the loose intuitions that had been sufficient for the generations prior. This led Georg Cantor (March 3, 1845 - January 6, 1918) to create his own definition of real numbers during his investigations into trigonometric series. As Cantor continued his work in formal definitions of number systems, he slowly realized that there was good reason to extend the real numbers beyond their traditional confines into what are now known as *transfinite numbers*. In doing this, Cantor set down a road, which, while revealing to him that some infinite sets were bigger than others, eventually drove him to be institutionalized.

In the discussion that follows, we will provide an exposition on the "sizes" of many familiar sets. In this context, we will use cardinality as our measure of size. Specifically, we will establish that $\mathbb{N}$ is isomorphic to $\mathbb{Q}$, and thus the two have equal cardinality. After this, we present some of the most important steps in constructing $\mathbb{R}$ via Dedekind cuts (we will not address Cantor's construction of $\mathbb{R}$) and show that $|\mathbb{R}| > |\mathbb{N}|$, i.e. the cardinality of the reals is greater than that of the naturals. We will then explore Cantor's transfinite set theory via the construction of transfinite numbers along with some of the philosophical implications he found therein. We will conclude with some insights into the development of another notions related to infinity, particularly the infinitesimal numbers. There will be an exposition of Cantor's philosophy of infinitesimal numbers along with a look at other developments that occurred both during and after his lifetime. Note that for the duration of this paper, we will define the natural numbers by the counting numbers; that is $\mathbb{N} \equiv \{1, 2, 3, \dots\}$. We also assume the Axiom of Choice.

2 Countable Sets

One of the most fundamental infinite sets is the set of natural numbers $\mathbb{N}$. Although one can construct $\mathbb{N}$ as sets of sets (as we will discuss later in this work), it is valid to consider the natural numbers the primitive numbers off of which all other (common) number systems are understood. Since the natural numbers have a definite order and are **nowhere dense** (i.e. over any finite interval, there are only finitely many natural numbers), we consider $\mathbb{N}$ to be a countable set. When we say countable, we do not wish that the reader envisions sitting to count all of the natural numbers, and one day finishing. Since $\mathbb{N}$ is infinite, this would be impossible, but that does not mean that there is not some sensible way to go about counting them if one were to try. The natural numbers are inherently enumerated since they are the numbers we use to count things, which is why they are considered countably infinite. The question arises as to whether all sets are countably infinite and how do we find out if they are. This is the question we explore below.

2.1 The size of $\mathbb{N} \times \mathbb{N}$

With Cantor's definition of a countable set in mind, it will be enlightening to establish some interesting consequences of this simple statement. We will first review some basic definitions as the proofs to follow will rely heavily on their consequences.

Definition 1: Let X and Y both be sets, and let f be some function which maps from X into Y. Then,we define f to be **injective** if $(\forall x_1, x_2 \in X)(f(x_1) = f(x_2) \to x_1 = x_2)$.

Definition 2: Let X and Y both be sets, and let $f : X \to Y$. Then we define f to be **surjective** if $(\forall y \in Y)(\exists x \in X)$ such that $(f(x) = y)$.

For injectivity, if the image of two elements of X are equal, then the elements themselves must be equal. Note that it is equivalent to say $x_1 \neq x_2 \to f(x_1) \neq f(x_1)$. If a function is surjective, we can say that every element in Y has some element in X which maps to it. Equivalently, every element in the codomain (Y) is in the image of f, where the **image** of f, denoted $\text{Im}(f) = \{f(x) \in Y | x \in X\}$.

The first two definitions are basic properties that f may or may not possess. Every function, however, has an image. In fact, if we restrict f such that $f : x \to \text{Im}(x)$, then f must be surjective. We can use these properties of f in order say something about the sets themselves.

Definition 3: Let X and Y be sets. Then, X is **isomorphic** to Y, denoted $X \cong Y$, if $(\exists f : X \to Y)$ such that f is injective and surjective.

Our next definition is a property of sets of integers.

Definition 4: Let $X \subseteq \mathbb{N}$ be non-empty. Then, by the **Well-Ordering Principle** $(\exists x \in X)$ such that (x is the least element of X).

We now have everything we need to proceed. First, we will show that any infinite subset of $\mathbb{N}$ must be countable in size. This is straightforward if it is assumed that countable is the only (or at least smallest) form of infinity. Here, we do not make that assumption.

Theorem 1 Every infinite subset of $\mathbb{N}$ is countable.

Let $A_1 \subseteq \mathbb{N}$ be infinite. We will show that A_1 is countable. We shall construct a sequence S of the elements of A_1. Define the elements of S as follows: By the Well-Ordering Principle, $\exists a \in A_1$ such that a is the least-element of A_1. Let $s_1 = a$ and $A_2 = A_1 \setminus \{a\}$. Now, A_2 is still a non-empty subset of $\mathbb{N}$, so the WOP applies. Then, $\exists b \in A_2$ such that b is the least element of A_2. Let $s_2 = b$ and $A_3 = A_2 \setminus \{b\}$. Repeat this procedure to generate S, which is an infinite ordered sequence of the elements of A_1. As S is ordered, we can define a function $f : \mathbb{N} \to A_1$ such that $f(i) = s_i$. We will show that f is surjective by contradiction. Suppose that f is not surjective. Then, $(\exists s \in A_1) st (\forall x \in \mathbb{N})(f(x) \neq s)$, ie s is not in the sequence. By construction, $s_1 < s_2 < s_3 < \cdots < s_n$. So, $s_1 < s_2 < s_3 < \cdots < s_n < \cdots < s$. However, $s \in A_1 \subseteq \mathbb{N}$, so s must be a finite natural number. Therefore, s has only finitely many predecessors $1, 2, \ldots, s-1$. As A_1 is infinite, there must be some A_n such that s is the least element. Therefore, $s \in S$, which contradicts our state that s was not in the sequence. Thus, f must be surjective. So, every element in A_1 must correspond with some element in $\mathbb{N}$, so they must be the same size. Therefore, A_1 must be countable. ■

This result may or may not surprise the reader, but it is very important in establishing our next theorem: If a map from a countable set to an infinite set is surjective, then the infinite set must in fact be countable. This is not as straightforward, but we will attempt to impose some intuition into it. If every item from the infinite set is mapped to by an element in the countable set, then in a loose sense, the infinite set can be associated with a subset of the countable set, which then implies that the infinite set must be countable. We put this argument on firm logical footing.

Theorem 2 Let X be countable, Y be infinite, and $f : X \to Y$ be surjective. Then Y is countable.

As f is surjective, $(\forall y \in Y)(\exists x \in X)st(f(x) = y)$. Define $g : Y \to X$ by $g(y) = x_y$, where $x_y \in X$ is chosen and guaranteed to exist by surjectivity. Now, g must be injective. If g were not injective, then two different y could be mapped to by the same x by f. This would contradict f being a function. As X is countable and $g(Y) \subseteq X$, by Theorem 1, we have that $g(Y)$ must be countable. So, as the image of g must be countable, and g is injective, $g(Y) \cong Y$. Therefore, Y must be countable. ∎

Finally, our last result in this section is perhaps the most surprising: $\mathbb{N} \times \mathbb{N}$ is countably infinite! Intuitively, we would expect this to be false. After all, we are taking two full instances of $\mathbb{N}$ and creating the set of *all possible pairs* from their elements. This would seem to suggest that there must be more elements in their Cartesian product, but, as we will see below, this is simply not true.

Theorem 3 $\mathbb{N} \times \mathbb{N}$ is countable.

We will show that $\mathbb{N} \times \mathbb{N}$ is countable.
Let $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ by $f(x, y) = \frac{1}{2}(x + y)(x + y + 1) + y$.
We will show that f is injective.
Let $(x, y), (x', y') \in \mathbb{N} \times \mathbb{N}$ such that $f(x, y) = f(x', y')$.
Then, $\frac{1}{2}(x + y)(x + y + 1) + y = \frac{1}{2}(x' + y')(x' + y' + 1) + y'$.
Assume for contradiction that $x + y \neq x' + y'$.
We will prove that $x + y < x' + y'$ produces a contraction.
The case of $x + y > x' + y'$ is analogous.
Therefore, $x = x'$ and thus $(x, y) = (x', y')$.
Assume $x + y \neq x' + y'$.
Assume $x + y < x' + y'$. The reverse case will be analogous.
Then, $x + y + 1 \leq x' + y'$.
Then, $f(x, y) = \frac{1}{2}(x + y)(x + y + 1) + y \leq \frac{1}{2}(x + y)(x' + y') + y$ as we are multiplying by a positive (potentially larger) number.
Now, $\frac{1}{2}(x + y)(x' + y') + y = \frac{1}{2}(x + y)(x' + y' + 1) - \frac{1}{2}(x + y) + y = \frac{1}{2}(x + y + 1)(x' + y' + 1) - \frac{1}{2}(x + y) - \frac{1}{2}(x' + y' + 1) + y$.
Then, $\frac{1}{2}(x + y + 1)(x' + y' + 1) - \frac{1}{2}(x + y) - \frac{1}{2}(x' + y' + 1) + y \leq \frac{1}{2}(x' + y')(x' + y' + 1) - \frac{1}{2}(x + y) - \frac{1}{2}(x' + y') - \frac{1}{2} + y$.
Then, $\frac{1}{2}(x' + y')(x' + y' + 1) - \frac{1}{2}(x + y) - \frac{1}{2}(x' + y') - \frac{1}{2} + y < \frac{1}{2}(x' + y')(x' + y' + 1) - (x + y) + y$.
So, $\frac{1}{2}(x' + y')(x' + y' + 1) - (x + y) + y = \frac{1}{2}(x' + y')(x' + y' + 1) - x < \frac{1}{2}(x' + y')(x' + y' + 1) + y' = f(x', y')$.
Therefore, we have shown $f(x, y) < f(x', y')$, which contradicts our assumption that $f(x, y) = f(x', y')$!
So, $x + y = x' + y'$.
By our assumption, $\frac{1}{2}(x + y)(x + y + 1) + y = \frac{1}{2}(x + y)(x + y + 1) + y'$.
Therefore, $y = y'$.
So, $x + y = x' + y$.

Therefore, $(x, y) = (x', y')$, so f must be injective.
Let $I = \text{Im}(f)$ which is a subset of $\mathbb{N}$.
Then, we can define $g : I \to \mathbb{N} \times \mathbb{N}$ by $g(i) = f^{-1}(i)$.
This must be surjective as f is injective, and we are only mapping from the image of f.
Then, by Theorem 1, I is countable as $I \subseteq \mathbb{N}$.
As such, by Theorem 2, $\mathbb{N} \times \mathbb{N}$ is countable. ■

This might seem like a lot of hand waving, and non-reflective of the reality of the situation, but that is simply not true. We created a function, which, although not injective over $\mathbb{R}$, is injective over the natural numbers. Many of the steps utilize the fact that we are working with only $\mathbb{N}$. For example, we took the fact that $x+y < x'+y'$ and deduced that $x+y+1 \leq x'+y'$. This is only true because there is no number between $x+y$ and $x+y+1$ in $\mathbb{N}$. If we were in $\mathbb{R}$, this would be false (for example, $x+y = 3.5$ and $x'+y' = 4$ so $x+y < x'+y' \not\to x+y+1 \leq x'+y'$). This observation is what allowed the proof to work. After defining f, we took its inverse, which is defined since the function is injective. We could then take this inverse function and apply the previous results, which allowed us to show that $\mathbb{N} \times \mathbb{N}$ is countable!

2.2 More on Countable Sets

We will now produce two more sets which are also countable. We will first show that the union of countably many countable sets is itself countable. We will then conclude with a proof that the rational numbers $\mathbb{Q}$ are also countable. The result of this can be summed up idiomatically by the idea that any operation applied to countable sets countably many times will always produce a countable set.

Theorem 4 A countable union of countably many infinite sets is countable.

Let X_i be a countable set for all $i \in I$, where I is a countable index set. Without loss of generality, we can assume that these sets are disjoint, ie $(\forall i, j \in I)(X_i \cap X_j = \emptyset)$. By the well-ordering principle, we can define an ordering for every set as exemplified in Theorem 1. We can then identify any element of any set as $x_{i,j}$ where i is the index value of the set, and j is the index value of the element in its respective set. Now, define $f : \mathbb{N} \times \mathbb{N} \to \bigcup_{i \in I} X_i$ as $f(i, j) = x_{i,j}$. This function is then trivially surjective as there are countably many sets (so there is no set which is not indexed) and each set is countable in size (so there is no value which is not indexed). By Theorem 2, as $\mathbb{N} \times \mathbb{N}$ is countable, and f is surjective, $\bigcup_{i \in I} X_i$ must be countable. ■

Theorem 5 $\mathbb{Q}$ is countable.

Consider $\mathbb{Q}^+ = \{p/q | p, q \in \mathbb{N}\}$. Let $f : \mathbb{N} \times \mathbb{N} \to \mathbb{Q}^+$ by $f(p, q) = p/q$. Since the elements of $\mathbb{Q}^+$ are defined as p/q, f must be surjective. Then, by Theorem 2, $\mathbb{Q}^+$ is countable. Symmetrically, $\mathbb{Q}^-$ defined by $\{-p/q | p, q \in \mathbb{N}\}$ must also be countable. Now, $\mathbb{Q} = \mathbb{Q}^+ \cup \{0\} \cup \mathbb{Q}^-$. Then, by the inductive proof used in Theorem 4, this union must be countable. ■

The notion that $\mathbb{Q}$ was countable came as quite a shock to Cantor. In some intuitive sense, the rationals are more dense than the natural numbers. After all, one can define an interval small enough that it contains no natural numbers (perhaps [0.25,0.75]), but this cannot be done for the rational numbers. Between any two rational numbers, there exist infinitely many more rational numbers. Another way to say this is that $\mathbb{Q}$ is **everywhere dense**. For this reason, it is reasonable to deduce that $\mathbb{Q}$ should, in some way, be bigger than $\mathbb{N}$. However, since we can create $\mathbb{Q}$ as a union of sets equal in size to $\mathbb{N} \times \mathbb{N}$, this must not be true.

It is tempting to think at this point that all infinite sets must be isomorphic to the naturals as all of our intuitions thus far have shown to lead us to false conclusions. However, in the next section we will show that the most common number system is uncountable, and, in fact, it is bigger than $\mathbb{N}$! These numbers we are speaking of are the real numbers, $\mathbb{R}$.

3 Uncountable Sets

Before discussing the size of the real numbers $\mathbb{R}$, we need some meaningful definition of what we mean by real numbers. Intuitively, we know what one means when they discuss a *real number*; it is any number which can be represented as a decimal expansion of the digits 0-9. This simplistic definition, however, is flawed because numbers do not have unique representations. Consider, for example, the expansion $0.\bar{9}$ and $1.\bar{0}$. These are obviously distinct decimal expansions, but, as you may be aware, they represent the same value! In fact, for infinitely many rational number, there are two unique decimal expansions. (Any number of the form $p/(2m + 5n)$, where p and *2m+5n* have no common factors, and $m, n \in \mathbb{N}$ is in fact such a number.) So, we need some better formal definition, which is where *Dedekind cuts* come into play.

3.1 Constructing $\mathbb{R}$

We will consider the rational numbers $\mathbb{Q}$ as partitioned into two sets A and B (i.e $A \cap B = \emptyset$ and $A \cup B = \mathbb{Q}$) such that A and B are both non-empty, A is closed downward [i.e. $(\forall x \in \mathbb{Q})(\forall a \in A)(x \leq a \to x \in A)$], B is closed upwards [i.e. $(\forall x \in \mathbb{Q})(\forall b \in B)(x \geq b \to x \in B)$], and A does not contain a greatest element. This constitutes a *Dedekind Cut*. By construction, B is completely determined by A, and $(\forall a \in A)(\forall b \in B)(a < b)$. We will refer to a Dedekind cut by the ordered pair of sets (A, B). We can then associate any real number r with some pair (A, B), which contains all rational numbers less than r.

Example: Define $A = \{q \in \mathbb{Q} | q < 0\} \cup \{q \in \mathbb{Q} | q \geq 0 \ \& \ q^2 < 2\}$. Then, (A, B) defines a Dedekind cut.

Define A as explained above. for (A, B) to be a Dedekind cut, it follows that $B = \mathbb{Q} \setminus A$. We must show that A is closed downward, B is closed upward, and A contains no greatest element. Let $x \in \mathbb{Q}$ and $a \in A$. Assume $x \leq a$. Case 1: Assume $x \leq 0$. If $x \leq 0$, then $x \in A$. Case 2: Assume $x > 0$. Then, we know $a > 0$ as $a > x > 0$. It therefore follows that $a^2 > x^2$. As $a \in A, x^2 < a^2 < 2$. So, $x^2 < 2$. Therefore, $x \in A$. So, A is downward closed. We must now show that B is upward closed. Let $x \in \mathbb{Q}$ and $b \in B$ such that $x \geq b$. As $x \geq b > 0, x^2 > b^2 > 2$. Thus, $x^2 > 2$ so $x \in B$. Therefore, B is closed upward. We must now show that A does not have a greatest element, ie $(\forall a \in A)(\exists x \in A)$ such that $(x > a)$. Let $a \in A$. Case 1: Assume $a \leq 0$. Let $x = 1$. Then, $x > a$. Case 2: Assume $a > 0$. Define $x = \dfrac{2a+2}{a+2}$. First, we must show that $x \in A$ by showing that $x \in \mathbb{Q}^+$ and $x^2 < 2$. As $a \in A$and $a > 0, a \in \mathbb{Q}^+$. Therefore, $a = p/q$ for some $p, q \in \mathbb{N}$. Then, $x = \dfrac{2p/q + 2}{p/q + 2} = \dfrac{\frac{2p+2q}{q}}{\frac{p+2q}{q}} = \dfrac{2p + 2q}{p + 2q} \in \mathbb{Q}^+$.

Moreover, $x^2 = \dfrac{4a^2+8a+4}{a^2+4a+4} < \dfrac{4(2)+8a+4}{(2)+4a+4} = \dfrac{4(3+2a)}{2(3+2a)} = 2$. Therefore, $x \in A$. Finally, as $a^2 < 2, a^2 + 2a < 2 + 2a$. Therefore, $a(a+2) < 2 + 2a$. As $a + 2 > 0$, this implies $a < \dfrac{2a+2}{a+2} = x$. Therefore, $x > a$. Therefore, A has no greatest element. Therefore, (A, B) defines a Dedekind cut. ■

It is beyond the scope of this discussion to show that this cut is associated with $\sqrt{2}$, but know that this is true. We have thus defined a Dedekind Cut and provided one example of a valid cut. Notice that we can construct both rational numbers and irrational numbers using this method. For example, we can construct 2 by defining $A = \{q \in \mathbb{Q} | q < 2\}$. If we consider $\mathcal{R}$ to be the set of all Dedekind cuts then we can see that $\mathbb{Q} \subset \mathcal{R}$ since we can construct all rationals using this method as well as numbers such as $\sqrt{2}$, which is not rational. One formality we must address before we can finish this discussion is the total-ordering of $\mathcal{R}$.

Theorem 6 Let $x = (A, B)$ and $y = (C, D)$ be Dedekind cuts. If we define $x \leq y$ as an ordering established by $A \subseteq C$, then $\leq$ defines a total-ordering.

Let $x = (A, B), y = (C, D)$, and $z = (E, F)$ be Dedekind Cuts. Then, the relation is antisymmetric iff $(x \leq y \ \& \ y \leq x \to x = y) \iff (A \subseteq C \ \& \ C \subseteq A \to A = C)$. The relationship is transitive iff $(x \leq y \ \& \ y \leq z \to x \leq z) \iff (A \subseteq C \ \& \ C \subseteq E \to A \subseteq E)$. Finally, the relationship is total iff $(x \leq y \text{ or } y \leq x) \iff (A \subseteq C \text{ or } C \subseteq A)$. The first two statements are trivially true since they are true of sets in general. Totality, however, is not generally true of sets. We will prove this via contradiction. So, assume $A \nsubseteq C$ and $C \nsubseteq A$. Let $a \in A \setminus C$ and $c \in C \setminus A$. Without loss of generality, assume $a < c$. Then, as C is closed downward, $a \in \mathbb{Q}$, and $a < c, a \in C$. This however, contradicts our assumption that $a \in A \setminus C$. Therefore, it must be true that the ordering is total. Therefore, $x \leq y \iff A \subseteq C$ defines a total-ordering. ■

Having thus shown that $\mathcal{R}$ is totally order, we propose that $\mathcal{R}$ can be used as a formal definition of $\mathbb{R}$. We will not prove this as it requires showing that $\mathcal{R}$ has all of the algebraic properties of $\mathbb{R}$ such as $(\mathcal{R}, +, \cdot)$ is a field (with some sensible definition of $+$ and $\cdot$), which is well beyond the scope of this paper. We do, however, invite the reader to explore any doubts they may have using the tools described herein. With a solid foundation for these familiar numbers now established, we will investigate their size.

3.2 The Cardinality of $\mathbb{R}$

So far, we have only considered sets which are isomorphic to the natural numbers, but as we now begin to explore sets of different "sizes" it will be useful to formally state what is meant by the "size" of a set. In various sub-fields of mathematics, this can mean a variety of things. In real analysis, one may be referring to the measure of a set, while in geometry one may be referring to the area of a surface. Here, when we speak of the "size" of a set, we are referring to its cardinality.

Definition 5: Let X, Y be non-empty sets. Then, the **cardinality** of X, denoted $|X|$, is equal to the cardinality of Y if there exists a bijective function $f : X \to Y$. That is, $|X| = |Y| \iff X \cong Y$. Moreover, $|X| < |Y|$ if there exists an injective function $f : X \to Y$, but no such surjective function exists.

Definition 6: Let X be a non-empty set. Then X is **uncountable** if $|X| > |\mathbb{N}|$.

It follows from Theorem 1 that $\mathbb{N}$ is the smallest infinite cardinality since any subset of $\mathbb{N}$ is either finite or isomorphic to $\mathbb{N}$. Thus, we can meaningfully say that $|\mathbb{N}| = \aleph_0$, with the null subscript suggesting that $\mathbb{N}$ is the smallest possible size that an infinite set can hold. It is also true that the cardinality of infinite sets is discrete; that is, $\aleph_{j+1}$ is the next largest cardinality after $\aleph_j$. The symbol we have used here $\aleph$ is the first letter of the Hebrew alphabet, aleph. This is the symbol chosen by Cantor to denote the size of infinite sets, and many believe it was due to the religious connotations behind the letter that he chose this symbol. In the Jewish mystic tradition known as Kabbalah, $\aleph$ is the symbol used to denote the transcendent aspect of God: the aspect of God that is inherently beyond the capacity of human beings to ever understand. Later in Cantor's life, he became quite religious, and so, out of a respect of sorts, many believe he chose $\aleph$ as a sign of reverence to God who was the Absolute Infinite.

In order to establish the fact that $\mathbb{R}$ must be uncountable, we will reproduce the traditional proof concerning whether the open interval $(0, 1)$ is countable. In introductory texts, this proof ends here, but we will show that $(0, 1) \cong (1, \infty)$. We could then consider $\mathbb{R} = (-\infty, -1) \cup \{-1\} \cup (-1, 0) \cup \{0\} \cup (0, 1) \cup \{1\} \cup (1, \infty)$. We will leave it to the reader to show that this union is the same cardinality as $(0, 1)$.

Theorem 7 The open interval $(0, 1)$ on the real number line is uncountable.

Assume for contradiction that $(0, 1)$ is countable. Then, we can construct an enumerated list of all the elements of the interval. Assume such a list exists where the elements are written as an infinite decimal expansion. (If the number terminates, such as 0.25, it will be written as $0.2500\bar{0}$). Then,

$$\begin{aligned}
a_1 &= 0.a_{11}a_{12}a_{13}\dots a_{1j}\dots \\
a_2 &= 0.a_{21}a_{22}a_{23}\dots a_{2j}\dots \\
a_3 &= 0.a_{31}a_{32}a_{33}\dots a_{3j}\dots \\
&\vdots \\
a_i &= 0.a_{i1}a_{i2}a_{i3}\dots a_{ij}\dots \\
&\vdots
\end{aligned}$$

where $a_{ij} \in \{0, 1, 2, \dots, 9\}$ so that any digit of any element can be identified by its indices. Consider the element a^* constructed $a_j^* = \begin{cases} a_{jj} + 1 & a_{jj} \neq 9 \\ 0 & a_{jj} = 9 \end{cases}$ Now, $a^* \in [0, 1]$ since it is still an infinite decimal expansion of a number between 0 and 1. So, $(\exists a_j)$ such that $(a^* = a_j)$. $a^* \neq a_1$ as they are not equal in their first index value. Likewise, $a^* \neq a_2$ as they are not equal in their second index value. In fact, by construction, every a_j disagrees with a^* at its j-th index. Therefore, a^* is not in the list, which contradicts the fact that the list was supposed to contain all real numbers in $(0, 1)$. Therefore, $(0, 1)$ cannot be countable. ■

Theorem 8 $(0, 1) \cong (1, \infty)$

Let $f(x) = x^{-1}$. Let $x, y \in (0, 1)$. Assume $f(x) = f(y)$. Then, $x^{-1} = y^{-1}$. Multiplying both sides by xy we find $y = x$. Therefore, f is injective. Let $c \in (1, \infty)$. Let $a = c^{-1}$. Then,

$f(a) = f(c^{-1}) = (c^{-1})^{-1} = c^{-1\cdot -1} = c$. Therefore, f is surjective. Thus, f is bijective. Therefore, $(0,1) \cong (1,\infty)$.$\blacksquare$

By the definition of cardinality, since there exists an injective function from $\mathbb{N} \to (0,1)$ (e.g. $f : \mathbb{N} \to (0,1)$ by $f(n) = \begin{cases} n^{-1} & n \neq 1 \\ 1/\pi & n = 1 \end{cases}$), but there does not exist a bijective function between the two (otherwise they would be isomorphic), then it follows that the cardinality of $(0,1)$ must be strictly larger than that of the natural numbers. This may come as a surprise to a reader who has not seen these results before; there are in fact "more numbers" between 0 and 1 than there are counting numbers. Despite the fact that $(0,1)$ is a finite length, it contains more elements (in $\mathbb{R}$) than there are rational numbers in the entire continuum.

You may have noticed that we have only established that $|\mathbb{R}| > |\mathbb{N}|$, but we have not given an aleph assignment to the real numbers. It is tempting to hypothesize that $|\mathbb{R}| = \aleph_1$, as the **Continuum Hypothesis** states as Cantor himself postulated. Throughout the rest of his life, Cantor made unsuccessful attempt after unsuccessful attempt to prove his hypothesis, but to no avail. Until the the very end, Cantor was tormented by this very question, which was the ultimate source of his mental destabilization. In 1940, Kurt Godël proved that the Continuum Hypothesis cannot be disproved from the axioms of Zermelo-Frankel Set Theory with Axiom of Choice(ZFC), which is the system in which we are inherently working. This might seem to suggest that the statement is true, but in 1963, Paul Cohen proved that the statement cannot be *proven* from ZFC! Therefore, assuming ZFC is consistent (which we have reason to believe that it is), it is beyond the scope of set theory itself to answer this question! All that we know is that $|\mathbb{R}| > |\mathbb{N}|$ but not how much bigger.

Two questions naturally arise from these results: are there sets with cardinality between $\mathbb{N}$ and $\mathbb{R}$, and are there sets with cardinality greater than $\mathbb{R}$? We know from the improvable nature of the Continuum Hypothesis that the first question is unanswerable, but we can make some headway into answering the second question. In the same way that we took the limit cases of the squares of rational numbers approaching 2 to define $\sqrt{2}$, we can take the limit case of natural numbers to define a new number ω, which opens us to the world of Transfinite Numbers $\mathbb{T}$.

4 Transfinite Set Theory

Before we begin investigating transfinite set theory, we wanted to clarify a few points about the topics which will be discussed below. We have given the symbol $\mathbb{T}$ to transfinite numbers, but this is not meant to suggest that there exists some grand set (or even class) which contains *every* transfinite number. Such a thing is forbidden in ZFC. One can loosely consider $\mathbb{T}$ as the collection of all number classes (defined below), and as such it is itself a class. However, one can not ask questions such as "what is the size of $\mathbb{T}$" as this is a meaningless question. An alternative, perhaps more helpful, way to think of $\mathbb{T}$ is that it is the collection of objects which results from Cantor's first and second principles of generation (defined below). We call this a collection in order to avoid any association of $\mathbb{T}$ with a set. With that said, let's begin our investigation into the transfinite numbers!

4.1 Constructing $\mathbb{T}$

In considering the creation of the transfinite numbers $\mathbb{T}$, Cantor reflected on the process by which $\mathbb{N}$ was generated. He considered this a "principle of adding units" and called it the **first principle**

of generation. Today, we may think of this as the successor function, but regardless of its name, the first principle is responsible for creating $\mathbb{N}$. This principle has no reason to presume an end, so there is no end to $\mathbb{N}$, which is why it is infinite. Cantor, however, considered a **second principle of generation** wherein any sequence of numbers with no largest value can then define a next number which is the value that comes after all of the numbers in the sequence. This process is *identical* to the structure of A in our example presented for Dedekind cuts. There is no largest rational number in A, and so we can define some new number which is the next value that supersedes all of the values in A (in the case of our example $\sqrt{2}$). Obviously, the number cannot be part of the sequence, otherwise it would define a largest value, so it must be a part of some larger superset ($\sqrt{2}$ is not in $\mathbb{Q}$ but it is in $\mathbb{R}$ which contains $\mathbb{Q}$).

When Cantor applied this method to the natural numbers, he defined ω as the first number created by the second principle such that $(\forall a \in \mathbb{N})(a < \omega)$. Then, by the first principle, there must be a value $\omega + 1$ since we now have a new base onto which we may add new units. The notation of what exactly $\omega + 1$ represents may be a bit confusing, so we will take a short aside to more fully understand the first principle of generation. One of the most common ways to construct $\mathbb{N}$ is via the successor function, $S(n) = \{n\}$. If we define $0 = \emptyset$, then $S(0) = \{\emptyset\}$, which we will call 1, and $S(1) = \{1\} = \{\{\emptyset\}\}$, which we can call 2! The second principle of generation essentially gave us a new starting point $\omega = \mathbb{N}$. Then, $S(\omega) = \{\omega\} = \{\mathbb{N}\} = \omega + 1$. In the interpretation of numbers themselves as simply sets of sets, this is a perfectly reasonable approach.

The application of the first and second principles of generation are both without end. Thus, in $\mathbb{T}$ we have $1, 2, 3, \ldots, \omega, \omega + 1, \omega + 2, \ldots, 2\omega, 2\omega + 1, \ldots$ And so, in a very natural manner, Cantor extended the **ordinal numbers**, or the numbers used to identify a specific item in a list. (Note: Ordinal is derived from the word "order." So, the ordinal numbers are the numbers used to specify an ordering of numbers.)

In the limit of this construction, one encounters a "natural break" in these new numbers. Just as ω, the first infinite number, defined the limit of the finite numbers, we may now define Ω as the limit to the countably infinite numbers, and thus as the first uncountably infinite number. So, this produces $1, 2, \ldots, \omega, \omega + 1, \ldots, \omega^{\omega}, \ldots, \omega^{\omega^{\omega}}, \ldots, \Omega, \ldots$. Cantor used this as inspiration for the **principle of limitation**, which defines a number class.

Definition 7: Let α be an ordinal, and $A = \{t \in \mathbb{T} \text{ such that } t < \alpha\}$. Then, $\{\alpha \in \mathbb{T} \text{ such that } |A| = k\}$ for some cardinal k defines a **number class**.

The first number class is $\mathbb{N}$, which has size $\aleph_0$. All ordinals from $[\omega, \Omega)$ define sets with cardinality $\aleph_0$ so this interval define the second class, which has size $\aleph_1$. For further discussion of Cantor's number classes, see [2].

4.2 Controversies and Implications

Cantor's work was not without controversy. Up until this development, infinity was seen as more of a variable than a number itself (as it is still usually still expressed in most mathematical curricula up through undergraduate). Infinity, as expressed by the symbol ∞, represented a *potential infinty* beyond all value and numerical meaning. Cantor believed, however, that this was not the only way to envision the infinite; instead he proposed a structure for the *actual infinite*, which was complete with mathematical soundness and meaning. But what exactly is the difference between ∞ and ω (and of course all the ordinals after ω)? Cantor argued that ∞ was a concept deeply embedded into the very heart of calculus. An ever-increasing collection of boxes which subdivided a space into sub-spaces of ever-decreasing area fundamentally underpins the idea of an integral. Here, the number of sub-spaces is never understood to be final; a space can *always* be further subdivided, and so this forms what Cantor called an *improper infinity.* The transfinite numbers, however, defined a new proper and complete form of infinity. ω *is* the (first) complete value that comes after every single natural number. There is no other thing that occupies this distinction, which makes ω a unique number just as 2 is the number that comes after 1 in the natural numbers.

As you might have expected, not everyone considered Cantor's distinction valid. Carl Gauss famously wrote to Heinrich Schumacher saying, "I protest above all against the use of an infinite quantity as a *completed* one, which in mathematics is never allowed. The infinite is only a *façon de parler*, in which one properly speaks of limits." Thus, the argument over the validity of Cantor's transfinite numbers was not a question of mathematical soundness, but of philosophical implication. Since the time of Aristotle, the general community rejected the concept of any complete infinity to avoid the paradoxes which followed therein. One can see, then, why Cantor's conscious reintroduction of complete infinities into the discussion provoked such dogmatic responses from across the disciplines. Cantor, however, believed in his mathematics not only on a conceptual level, but also on a philosophical level, so he went to great lengths to defend the system from attacks on any front.

Cantor believed that the heart of opposition lay in (false) assumption that the properties of the infinite reflect those of the finite. From Aristotle through to modern philosophers, this was the assumption which formed the basis of their objection, and so Cantor set out to refute it. As formulated by Aristotle, it is true for finite numbers that, given two real values a and b, $a+b > a$ and $a+b > b$. However, $a+\infty = \infty$, and so infinity seems to violate a fundamental property of addition. Cantor argued that it was naïve to presume that the properties of the finite fully determined those of the infinite. Moreover, under the formulation of $\mathbb{T}$ that Cantor created, $\omega + 1$ is in fact unique from ω even though both are infinite. What one loses is the commutative property of addition, $1 + \omega \neq \omega + 1$. In the first case, $1 + \omega = \omega$ since $1, 1, 2, \ldots$ is identical to $1, 2, 3, \ldots$. However, in the second case, $\omega + 1 \neq \omega$ since $1, 2, \ldots, 1$ is unique from $1, 2, 3, \ldots$. Were finite numbers to *never* affect the infinite, then this would seriously undermine any meaningful definition of addition, and Aristotle's views would be well-founded. However, in Cantor's formulation, the instances in which the finite affect the infinite are merely restricted with respect to the instances whereby a finite value can affect another finite value.

Having addressed the common error behind rejecting the actual infinite, Cantor goes to great lengths to distinguish between the real numbers $\mathbb{R}$ and *real* numbers, numbers which actually mapped onto reality in some meaningful way. To Cantor, the counting numbers were just as metaphysically real as $\mathbb{R}$, as were the transfinite numbers. When pressed for some application of his *real* transfinite numbers, Cantor asked the question of the cardinality of the set of "monads" (indivisible sub-components) which constituted all matter in the universe. Cantor postulated (without explanation) that this was $\aleph_0$, while the cardinality of the the set of monads constituting the aether was $\aleph_1$. Cantor also argued that the physical reality of $\mathbb{T}$ could be supported by its geometric representation. Just as π was given real significance as the ratio of the circumference of a circle to its diameter, so too were the transfinite numbers given meaning by the fact that the set of monads was in fact infinite. In so far as these numbers could be used to describe some aspect of physical reality, they must themselves have some objective existence.

Today, there is overwhelming evidence against the existence of the aether (although a small community of physicists are trying to give it a new validity), but the question as to the cardinality of the the set of all things that make up matter and energy is an interesting one. On the one hand, there is the tradition in science to attempt to break things down into their fundamental constituents, but throughout history this has proven time and again to fail us as we continue to discover further sub-units with scientific advancements. To further confound the question, we run into the difficulty that there is overwhelming evidence to suggest that the "number of monads" in the universe is not constant. If we accept the common postulate that dark energy exists in the universe, and the less common postulate that it is vacuum energy, this means that every second more energy is produced, which means (ultimately) more monads. Does this ultimately affect the cardinality of the set, or is the change such that we always remain in the same number class? On a smaller scale, we know that particles are created and destroyed on time-scales smaller than it is (fundamentally) possible to constrain. How then do we account for these monads? Thus, it is intrinsically impossible to take a snapshot of the universe and "count" all of the monads, as fundamental physics tells us that the uncertainty with respect to our count must be infinite. Nonetheless, it is reasonable to assume that these fluctuations cannot change the number class (and therefore the cardinality) of the set of monads in the universe, and so there must in fact be some cardinality to this set. Any speculation beyond this postulate is ultimately unfounded, and so it is irresponsible to actually prescribe a cardinality to the set.

Cantor also argued for the reality of $\mathbb{T}$ using the traditional arguments of finitists. Understanding that there was no end to the real number line, finitists allowed that for any arbitrarily large N, there exists a number n such that $n > N$. Cantor responded by concluding that this implicitly assumed the existence of all such n, the collection of which Cantor dubbed to be the *Transfinitum*. He then proceeded to argue that "Every potential infinity (the wandering limit) leads to a *Transfinitum* (the sure path for wandering), and cannot be thought of without the latter" when explaining an analogy of a wanderer taking a journey. In this, we see that Cantor understands $\mathbb{T}$ as essential to understanding the traditional ∞.

Along this line of thought, Cantor came to the surprising conclusion that the existence of irrational numbers, which were well accepted at the time, was equivalent to that of the transfinite. Irrational numbers can exist if and only if transfinite numbers exist. Cantor's construction of the irrational numbers is very similar to Dedekind cuts, which rely on the existence of a potentially infinite set. If A were not infinite in size, there would be a greatest element, which would contradict the construction of A. As argued above, where there is a potential infinity, there is a *Transfinitum*. Thus, one cannot construct the real numbers without the transfinite numbers. This led Cantor to assert that logically a mathematician could not accept the irrational numbers without the transfinite, and very few people were willing to give up the irrationals. Despite this argument, most mathematicians were still unwilling to accept the transfinite numbers.

5 Infinitesimal Numbers

5.1 Infinitesimal Numbers during Cantor's Life

In a review of Cantor's work on the transfinite numbers, Benno Kerry proposed the following formal definition for the infinitesimal numbers $\mathbb{E}$: "In my opinion a *formal definition* of definite,infinitely small numbers is indeed given in fixing the greatest of such numbers as one which produces the sum 1 by adding itself to itself ω times; the next smaller is then the one which produces 1 by adding itself to itself $\omega - 1$ times, etc. The definite, infinitely small numbers would be denoted as: $\frac{1}{\omega}, \frac{1}{\omega+1}, \ldots, \frac{1}{2\omega}, \ldots, \frac{1}{\omega^2}$ etc." In fact, many mathematicians argued that Cantor's transfinite number led naturally to such a definition of the infinitesimals. This, however, was completely out of the question for Cantor. In an argument strikingly similar to those lobbed against the transfinite numbers, Cantor claimed that any attempt to codify infinitesimals was unfounded and senseless.

He even went so far as to formulate his opposition in the form of a theorem.

Theorem 9 **Cantor's Infinitesimal Theorem** Non-zero linear numbers ξ (in short, numbers which may be thought of as bounded, continuous lengths of a straight line) which would be smaller than any arbitrarily small finite number do not exist, that is, they contradict the concept of linear numbers.

Cantor's proof was based on the Archimedean Axiom, which asserts that $(\forall a, b \in \mathbb{R} | a < b)(\exists n \in \mathbb{N})$ such that $(na > b)$. He argued that if we tried to extend this into the infinitesimal numbers, then there would never exist an $n \in \mathbb{T}$ to satisfy this condition. Since linear numbers are closed under multiplication, and $(\forall n \in \mathbb{T})(na \notin \mathbb{R})$, infinitesimals cannot be considered linear numbers and thus had no metaphysical meaning. Without metaphysical meaning, these were nothing more that "paper numbers." Despite entreatments and logical arguments to Cantor to see the infinitesimal as equally critical as the infinite numbers, Cantor remained stubborn due to his conviction that such numbers cannot exist due to the completeness of $\mathbb{R}$. Because he assumed a bijection from $\mathbb{R}$ to a geometric line, there was no room for infinitesimals. Ultimately, he saw the infinitesimals as inconsistent, and as such rejected them without question. Without physical correspondence, there was nothing to be understood about the infinitesimals. For further discussion on Cantor's life, work, and philosophy, see [1].

5.2 Developments of $\mathbb{E}$

The work developed herein is drawn primarily from [3]. Much to Cantor's dismay, most of the mathematical community did not agree with his outright objection to infinitesimals, and, hence, much has been developed over the years. Since its conception, mathematics has asked strikingly similar questions. In the birth of the irrationals, the Greeks asked what would satisfy the equation $x^2 = 2$ was. Later, mathematicians wondered what would be the effect if we defined a value such that $x^2 = -1$ had a solution, and thus arose the complex numbers. In constructing $\mathbb{E}$, we ask the question what would happen if $x^2 = 0$ were not simply solved by $x = 0$. We can then define $\mathbb{D} = \{\delta | \delta^2 = 0\}, \mathbb{E} = \{a + b\delta | a, b \in \mathbb{R} \ \& \ \delta \in \mathbb{D}\}$, and axiomatically establish that $(\forall f : \mathbb{E} \to \mathbb{R})(\exists$ a unique $b \in \mathbb{R})$ such that $(\forall \delta \in \mathbb{D})(f(x + \delta) = f(x) + b\delta)$. In is important to note that $\mathbb{D}$ must contain at least two elements. It is true that $0 \in \mathbb{D}$, but any $b \in \mathbb{R}$ satisfies $f(0 + 0) = f(0) + b \cdot 0$. So, there must exist at least one element δ_0 such that $\delta_0 \neq 0$ and $\delta_0^2 = 0$.

Just as with the transfinite numbers, we must sacrifice something in order to work in $\mathbb{E}$. For transfinite numbers, we lost commutativity of addition. Here, we lose the Law of the Excluded Middle, which we will define here as a number is either equal to zero or not equal to zero.

Theorem 10 The Law of the Excluded Middle is incompatible with the infinitesimal numbers.

Let $f : \mathbb{E} \to \mathbb{R}$ by $f(x) = \begin{cases} 1 & x \neq 0 \\ 0 & x = 0 \end{cases}$ This is an example of the Law of the Excluded Middle. Assume for contradiction that the two are compatible. Then, as we argued before, δ_0 must exist with some unique b for this function. Since $\delta_0 \neq 0, 1 = f(\delta_0) = f(0) + b\delta_0 = b\delta_0$. Then, squaring both sides yields $1 = b^2\delta_0^2 = b^2 \cdot 0 = 0$. This, however, is a contradiction. Therefore, the Law of the Excluded Middle does not exist in $\mathbb{E}$.$\blacksquare$

It is natural to wonder how to find this $b \in \mathbb{R}$.

Theorem 11 **Taylor's Formula** $(\forall f : \mathbb{E} \to \mathbb{R})(\forall x \in \mathbb{R})(\forall \delta \in \mathbb{D})(f(x+\delta) = f(x) + f'(x)\delta)$ where $f'(x)$ represents the derivative of f.

Example Use Taylor's Formula to show that $(f \cdot g)' = f' \cdot g + f \cdot g'$.

By Taylor's Formula, $(f \cdot g)'(x+\delta) = (f \cdot g)(x) + (f \cdot g)'\delta$.
We know, however, that

$$\begin{aligned}(f \cdot g)'(x+\delta) &= f(x+\delta) \cdot g(x+\delta) \\ &= (f(x) + f'(x)\delta) \cdot (g(x) + g'(x)\delta) \\ &= f(x)g(x) + f'(x)g(x)\delta + f(x)g'(x)\delta + f'(x)g'(x)\delta^2 \\ &= f(x)g(x) + f'(x)g(x)\delta + f(x)g'(x)\delta.\end{aligned}$$

So, $(f \cdot g)(x) + (f \cdot g)'(x)\delta = f(x)g(x) + f'(x)g(x)\delta + f(x)g'(x)\delta$. Therefore, $(f \cdot g)' = (f'(x)g(x) + f(x)g'(x))\delta$. As this relation is true for all $\delta \in \mathbb{D}$, we can drop the δ. Therefore, $(f \cdot g)'(x) = f'(x)g(x) + f(x)g'(x)$.$\blacksquare$

Notice that Taylor's Formula is in fact the first two terms of a Taylor series! If we define $d = \delta_1 + \delta_2$, then we can expand this to include the third term (thus defining the second derivative). If we want to continue this to higher orders, however, we need to generalize our set $\mathbb{D}$.

Definition 8: Let $\mathbb{D}_k = \{x | x^{k+1} = 0\}$. Then, $(\forall k \in \mathbb{N})(\forall f : \mathbb{D}_k \to \mathbb{R})(\exists$ a unique $b \in \mathbb{R})$ such that $(\forall \delta \in \mathbb{D}_k)\left(f(\delta) = f(0) + \sum_{i=1}^{k} b_i \delta^i\right)$.

With this generalization of $\mathbb{D}$, we can also extend $\mathbb{E}_k = \left\{a_0 + \sum_{i=1}^{k} a_i \delta^i | a_0, a_i \in \mathbb{R}\right\}$. Thus, although we will not, we now have the tools to prove $(\forall f : \mathbb{E}_k \to \mathbb{R})(\forall x \in \mathbb{R})(\forall \delta \in \mathbb{D}_k)$ $\left(f(x+\delta) = f(x) + \sum_{i=1}^{k} \frac{\delta^i}{i!} f^i(x)\right)$.

Although we have provided only the most cursory of overviews with respect to modern developments of infinitesimal numbers, it is clear that they are a fully developed extension of the number line with no less validity that Cantor's transfinite numbers. For readers interested in further exploring the construction and results of formal models, it is important to note that it is necessary to approach the topic using a new class of logic called *intuitionistic logic.* Moreover, if one is interested in exploring items of the form $x^k = 0$, we would point the reader to ring theory, where there is much work in *nilpotent* objects; for example, $\mathbf{x} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ is nilpotent as $\mathbf{x}^k = \mathbf{0}$ for all $k > 1$.

6 Conclusion

Without putting much thought into it, one might assume that all infinite sets were of equal size as measured by cardinality. With a little more thought, our hypothetical thinker might decide that there are in fact many sizes of infinity noting that there seem to be more rationals than naturals since the rationals are everywhere dense while the naturals are nowhere dense. As we have seen in this exposition, our thinker was on the right path to conclude that some infinities are larger than others, but the relative sizes of the sets come as quite a surprise. As it turns out, $\mathbb{N}, \mathbb{N} \times \mathbb{N}$, and $\mathbb{Q}$ are all equal in that they are all countable sets. The real number line $\mathbb{R}$, as we have seen, however, is intrinsically somehow different from these sets in that it is uncountable.

In an attempt to understand other sets larger than $\mathbb{N}$, we extended the natural numbers into the transfinite numbers $\mathbb{T}$. We did this by introducing a new principle of generation which defines a new number as the value that comes after an infinitely increasing set. In this, we created ω and then explained how this provided a new basis for generating numbers in the traditional fashion. When we noticed that a "special" occurrence of the new principle of generation occurred in the limit of countably infinite applications of the second principle, we gave this number a new name Ω, and it defined a new number class.

We then took some time to address the ways in which Cantor justified his new number system to mathematicians, theologians, and philosophers alike, from addressing fundamental errors in understanding the infinite numbers to arguing that accepting the irrational numbers was equivalent to accepting the transfinite, as one cannot exist without the other since their method of generation is nearly identical.

Finally, we looked at the "inverse" of the transfinite, the infinitesimals. Cantor was objectively opposed to the infinitesimals since they seemed to undermine the completeness of the reals, which he valued so strongly. In this, however, Cantor was wrong, and infinitesimals have since proven to be just as valid as the transfinite. We explored one of the constructions of the infinitesimals, which is intimately related to differentiation.

We hope that our survey of the sizes of infinity and the philosophical discussions inherent therein have proven insightful to the reader. The infinite is not a topic to be considered lightly, but we have attempted to make the subject as approachable as possible. In the end, we are forced to remember that many of our questions are simply unanswerable in the current formulation of set theory, but we surely hope that the reader has still found the subject engaging, worth while, and understandable. If we have succeeded in this, there is nothing more we can ask.

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