6-2017

An Investigation of the Four Vertex Theorem and its Converse

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Submitted in partial fulfillment

of the requirements for

Honors in the Department of Mathematics

UNION COLLEGE

March, 2017

Abstract

In the study of curves there are many interesting theorems. One such theorem is the four vertex theorem and its converse. The four vertex theorem says that any simple closed curve, other than a circle, must have four vertices. This means that the curvature of the curve must have at least four local maxima/minima. In my project I explore different proofs of the four vertex theorem and its history. I also look at a modified converse of the four vertex theorem which says that any continuous real-valued function on the circle that has at least two local maxima and two local minima is the curvature function of a simple closed curve in the plane. The modified converse has a rather long history and was only resolved in 1997.

1 Introduction

Curves have been studied by mathematicians for years. In the field of differential geometry there are many interesting results. One interesting result is the four vertex theorem and its converse. In the preceding sections I give an overview of the differential geometry needed to understand this theorem and a synopsis of various proofs of the theorem. I also look at a couple of interesting results based upon these works.

1.1 Parametrization

One way of defining curves is by parametrization. The definition of a parametrized curve is as follows,

Definition 1. *A curve in $\mathbb{R}^n$ is a map $\gamma : (\alpha, \beta) \to \mathbb{R}^n$, for some α, β with $-\infty \leq \alpha < \beta \leq \infty$.*

For our purposes we will be looking at plane curves, which are curves specifically in $\mathbb{R}^2$. A basic example of a parametrization is that of the circle.

Example 1. *A parametrization for a general circle can be given in the form*

$\gamma(t) = (a + r\cos(t), b + r\sin(t))$ *for* $0 \leq t \leq 2\pi$.

An example of a unit circle centered at the origin can be seen in figure 1
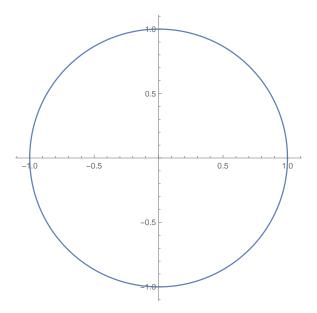

Figure 1: An example of a circle parametrized by the equation $\gamma(t) = (cos(t), sin(t))$

For a curve in $\mathbb{R}^2$ given by $\gamma(t) = (x(t), y(t))$ the derivative is defined to be $\dot{\gamma}(t) = (\dot{x}(t), \dot{y}(t)$. A curve is called smooth if all derivatives of arbitrary high order exist. We will work with curves that we are assuming to be smooth.

Once you have a parametrized curve you can define the arc length of the curve starting at the point $\gamma(t_0)$ as

$$s(t) = \int_{t_0}^{t} \|\dot{\gamma}(u)\| du$$

Another important term to define is a unit speed curve.

Definition 2. *A unit speed curve is a curve,* $\gamma : (\alpha, \beta) \to \mathbb{R}^n$, *such that* $\dot{\gamma}(t)$ *is a unit vector for all* $t \in (\alpha, \beta)$. *That is* $\|\dot{\gamma}(t)\| = 1$ *for all* $t \in (\alpha, \beta)$.

It turns out that curves are much easier to work with when they are unit speed, which is why this is an important concept.

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Another important fact about parametrizations is that they are not unique. One curve can be parametrized in many different ways.

For example consider the parametrization of a line given by

$$\gamma(t) = (5 + 2t, 7 + 18t)$$

this is equivalent to

$$\gamma(t) = (6 + t, 16 + 9t)$$

We define a reparametrization of a curve as follows

Definition 3. *A parameterized curve $\tilde{\gamma}(t) : (\alpha, \beta) \to \mathbb{R}^n$ is a reparametrization of a parametrized curve $\gamma(t) : (\alpha, \beta) \to \mathbb{R}^n$ if there is a smooth bijective map $\phi : (\tilde{\alpha}, \tilde{\beta}) \to (\alpha, \beta)$ such that the inverse map $\phi^{-1} : (\alpha, \beta) \to (\tilde{\alpha}, \tilde{\beta})$ is also smooth and $\tilde{\gamma}(\tilde{t}) = \gamma(\phi(\tilde{t}))$ for all $t \in (\tilde{\alpha}, \tilde{\beta})$.*

A reason why reparametrizations are important is that many curves can be reparametrized as unit speed curves. To explore this, we need to define what a regular curve is. A point on a curve $\gamma(t)$ is said to be a regular point if $\dot{\gamma}(t) \neq \vec{0}$. If $\dot{\gamma}(t) = \vec{0}$ for come t then the corresponding point is said to be a singular point. A curve is regular if all of its points are regular.

Now we can state the following proposition,

Proposition 1. *A parametrized curve has a unit speed reparametrization if and only if it is regular.*

Proof. The idea of the proof is as follows. First, let $\gamma(t) = (\gamma_1(t), \gamma_2(t))$ be a regular curve, then we define $s(t)$ as

$$s(t) = \int_{t_0}^{t} \|\dot{\gamma}(u)\| du$$

and then we note that

$$\dot{s}(t) = \sqrt{\dot{\gamma}_1(t)^2 + \dot{\gamma}_2(t)^2}$$

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We can then consider this to be the composition of two different functions f, g such that

$$f(t) = (\dot{\gamma}_1(t), \dot{\gamma}_2(t))$$

and

$$g(x_1, x_2) = \sqrt{x_1^2 + x_2^2}$$

So then it is clear that $g \circ f(t) = \dot{s}(t)$ and since f and g are both smooth, by the chain rule we know that $\dot{s}(t)$ is also smooth, and so therefore $s(t)$ is also smooth. Also, $\dot{s}(t) > 0$ since $\dot{s}(t) = \|\dot{\gamma}(t)\|$ and γ is regular.

This allows us to use the Inverse Function theorem to say that $s(t)$ has an inverse $t = \phi(s)$ and that this inverse is smooth. Now we will define $\psi(t) = s(t)$. So then,

$$\phi(\psi(t)) = t$$
$$\Rightarrow \frac{d}{dt}(\phi(\psi(t))) = 1$$
$$\Rightarrow \frac{d\phi}{ds}\frac{d\psi}{dt} = 1$$
$$\Rightarrow \frac{dt}{ds}\frac{ds}{dt} = 1$$
$$\Rightarrow \frac{dt}{ds} = \frac{1}{ds/dt} = \frac{1}{\|\dot{\gamma}(s)\|}$$

Now we can define a reparametrization $\tilde{\gamma}(s)$ as $\tilde{\gamma}(s) = \gamma(\phi(s))$. Then,

$$\frac{d\tilde{\gamma}}{ds} = \frac{d\gamma}{dt}\frac{dt}{ds}$$
$$= \frac{d\gamma/dt}{\|\dot{\gamma}(t)\|}$$
$$= \frac{\dot{\gamma}(t)}{\|\dot{\gamma}(t)\|}$$

So then it is easy to see that $\|\frac{d\tilde{\gamma}}{ds}\| = 1$, so that $\tilde{\gamma}$ is a unit speed reparametrization. $\qquad \square$

Another important concept is how we define what it means for a curve to close up.

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Obviously, a straight line will never close up, while a circle will always close up no matter which point you start at. We can define a meaning for the idea of a curve closing up as follows

Definition 4. *Let $\gamma : \mathbb{R} \to \mathbb{R}^n$ be a smooth curve and let $T \in \mathbb{R}$. We say that γ is T-periodic if $\gamma(t + T) = \gamma(t)$ for all $t \in \mathbb{R}$. If γ is not constant and is T-periodic for some $T \neq 0$, then γ is said to be closed.*

Another important type curve is simple curves.

Definition 5. *A curve is simple if it does not cross itself.*

Simple curves are easier to study since the fact that they do not cross themselves means that they do not have two different points that appear to be the same point.

Another important classification of curves that is important for us to define is convex curves, as we will be studying these in more detail later.

Definition 6. *A simple closed curve γ is said to be convex if its interior $int(\gamma)$ is convex, that is that the straight line joining any two points of $int(\gamma)$ is contained entirely in $int(\gamma)$.*

1.2   Curvature

Now we will look at the concept of curvature. Curvature is basically a measure of how curvy a given curve is. A more formal definition is as follows

Definition 7. *If γ is a unit speed curve with parameter t, its curvature $k(t)$ at the point $\gamma(t)$ is defined to be $\|\ddot{\gamma}(t)\|$.*

This shows why unit speed parametrizations are so important. Curvature can be defined for any regular curve in $\mathbb{R}^3$ as follows

$$k = \frac{\|\ddot{\gamma} \times \dot{\gamma}\|}{\|\dot{\gamma}\|^3}$$

which is obviously a much more complicated formula.

We can show that this formula reduces back down to $k = \|\ddot{\gamma}(t)\|$ in the case where $\|\dot{\gamma}\| = 1$. Since $\|\dot{\gamma}\| = 1$, we know $\|\dot{\gamma}\|^3 = 1$. Also we can rewrite $\|\ddot{\gamma} \times \dot{\gamma}\|$ as $\|\ddot{\gamma}\|\|\dot{\gamma}\| \sin(\theta)$ and since $\ddot{\gamma}$ and $\dot{\gamma}$ are perpendicular $\sin(\theta) = 1$, so we have

$$k = \frac{\|\ddot{\gamma}\| * 1 * 1}{1} = \|\ddot{\gamma}\|$$

If we limit ourselves to considering strictly plane curves we can give k a slightly more physical meaning and assign a sign to it. We start by defining $\mathbf{t} = \dot{\gamma}$, where we are assuming that γ is a unit speed curve, this is the tangent vector of γ. We can then define $\mathbf{n}_s$ to be the signed unit normal of γ, which comes from rotating $\mathbf{t}$ counterclockwise by $\frac{\pi}{2}$. If we define $\gamma(s) = \langle x(s), y(s) \rangle$ then

$$\mathbf{t} = \dot{\gamma}(s) = \langle \dot{x}(s), \dot{y}(s) \rangle \tag{1}$$

and

$$\mathbf{n}_s = \langle -\dot{y}(s), \dot{x}(s) \rangle. \tag{2}$$

Since $\mathbf{n}_s$ is perpendicular to $\mathbf{t}$, it is therefore parallel to $\dot{\mathbf{t}}$ and $\dot{\mathbf{t}} = \ddot{\gamma}$, so therefore we can find a scalar function of s, k_s, such that $\ddot{\gamma} = k_s \mathbf{n}_s$. We call this k_s the signed curvature of γ. In terms of the original function we can write this as

$$\ddot{\gamma} = k_s(-\dot{y}, \dot{x}) \tag{3}$$

It is important to note that it can be shown that

$$k_s = \frac{\dot{x}\ddot{y} - \ddot{x}\dot{y}}{((\dot{x})^2 + (\dot{y})^2)^{3/2}} \tag{4}$$

for a non unit speed curve.

The signed curvature, k_s, can be thought of geometrically as the rate at which the tangent

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vector rotates. If we have a unit speed curve γ, the direction of the of the tangent vector $\dot{\gamma}(s)$ is measured, a priori locally, by the angle $\phi(s)$ such that

$$\dot{\gamma}(s) = (\cos(\phi(s)), \sin(\phi(s))) \tag{5}$$

This leads us to our next proposition

Proposition 2. *Let $\gamma : (\alpha, \beta) \to \mathbb{R}^2$ be a unit speed curve, let $s_0 \in (\alpha, \beta)$ and let ϕ_0 be such that $\dot{\gamma}(s_0) = (\cos(\phi_0), \sin(\phi_0))$. Then there is a unique smooth function $\phi : (\alpha, \beta) \to \mathbb{R}$ such that $\phi(s_0) = \phi_0$ and that equation (5) holds for all $s \in (\alpha, \beta)$.*

The proof of this proposition will be omitted, but this proposition allows us to define the turning angle as follows

Definition 8. *The smooth function ϕ in Proposition 2 is called the turning angle of γ determined by the condition $\phi(s_0) = \phi_0$.*

Now we can state and prove the proposition that k_s is the rate at which the tangent vector rotates.

Proposition 3. *Let $\gamma(s)$ be a unit-speed plane curve, and let $\phi(s)$ be a turning angle for γ. Then,*

$$k_s = \frac{d\phi}{ds}$$

The proof of this is straightforward.

Proof. First, by equation (5) the tangent vector is

$$\mathbf{t} = (cos(\phi), sin(\phi))$$

so then,

$$\dot{\mathbf{t}} = \dot{\phi}(-sin(\phi), cos(\phi))$$

And since $\mathbf{n}_s = (-sin(\phi), cos(\phi))$, we can use the equation $\dot{\mathbf{t}} = k_s \mathbf{n}_s$ to get the result. $\quad\square$

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Now we will look at the Fundamental Theorem for curves in $\mathbb{R}^2$.

Theorem 1. *Let $k : (a, b) \to \mathbb{R}$ be any smooth function. Then, there is a unit-speed curve $\gamma : (\alpha, \beta) \to \mathbb{R}^2$ whose signed curvature is k. Further, if $\tilde{\gamma} : (\alpha, \beta) \to \mathbb{R}^2$ is any other unit-speed curve whose signed curvature is k, there is a direct isometry M of $\mathbb{R}^2$ such that*

$$\tilde{\gamma}(s) = M(\gamma(s)) \quad \forall s \in (\alpha, \beta).$$

In plain English this theorem is saying that planar regular curves are classified (up to rigid motions) by their signed curvature. The proof of the first part of this theorem is as follows,

Proof. Fix $s_0 \in (\alpha, \beta)$ and define, for any $s \in (\alpha, \beta)$, two functions $\phi(s)$ and $\gamma(s)$ as follows,

$$\phi(s) = \int_{s_0}^{s} k(u)du \tag{6}$$

and

$$\gamma(s) = \left(\int_{s_0}^{s} \cos(\phi(t)), \int_{s_0}^{s} \sin(\phi(t)) \right) \tag{7}$$

So then the tangent vector for γ is

$$\dot{\gamma}(s) = (\cos(\phi(s)), \sin(\phi(s)))$$

which is clearly a unit vector that makes an angle $\phi(s)$ with the x-axis. Thus γ is unit-speed, so by Proposition (3) its signed curvature is

$$\frac{d\phi}{ds} = \frac{d}{ds} \int_{s_0}^{s} k(u)du = k(s).$$

$\square$

We will use this technique later on.

2 Four Vertex Theorem

The statement of the four vertex theorem is very simple.

Theorem 2. *A simple closed curve in the plane, other than a circle, must have at least four vertices, that is, at least four points where the curvature has a local maximum or local minimum.*

A simple example is that of the ellipse, which has exactly four vertices.

Example 2. An ellipse can be parametrized by the following equation

$$\gamma(t) = (a\cos(t), b\sin(t)),$$

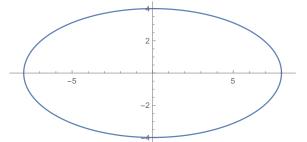the graph of an ellipse with $a = 8$ and $b = 4$ can be seen in figure 2.

Figure 2: The graph of $\gamma(t) = (8\cos(t), 4\sin(t))$.

the curvature $k(t)$ of an ellipse, which can be derived from equation (4) since the general parametrization of an ellipse is not a unit speed parametrization, is given by

$$k_s(t) = \frac{ab}{(a^2\cos^2(t) + b^2\sin^2(t))^3},$$

The graph of which can be seen in Figure 3. Four vertices, or two maximum and two minimum can clearly be seen in Figure 3 because the points 0 and 2π should be viewed as the same since the ellipse raps back around in a period of 2π.
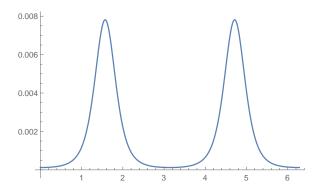
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Figure 3: The curvature of the ellipse in figure 2, given by the equation $k_s(t) = \frac{32}{(64\cos^2(t)+16\sin^2(t))^3}$.

The ellipse is the most simple case, having exactly four vertices, but it is easy to construct more complicated curves with many more vertices.

2.1 History

The four vertex theorem was first stated in 1909 by Syamadas Mukhopadhyaya. He also published the first proof of the theorem at the same time. He proved the four vertex theorem for convex curves only. Mukhopadhyaya published this proof very early on in his career, but he never went on to prove this theorem in general. The first general proof of the four vertex theorem was published in 1912 by Adolph Kneser. Kneser's proof of the theorem uses a projective argument. Kneser's son, Hellmuth, then published his own proof of the theorem in 1922. In 1944 Jackson published a proof by exclusion, where he categorized the curves that only have two vertices, rather than those with four or more.

Since these early proofs, many more mathematicians have come up with proofs of their own. We will examine two of them in further detail.

2.2 Osserman's Proof

The first proof we will look at is the one published by Robert Osserman in 1985 [1]. This proof is one of the more simple proofs of the four vertex theorem, and it works for any

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arbitrary simple closed curve, not just convex curves. Another difference is that Osserman's proof is a direct one, rather than a proof by contradiction.

Osserman's proof is based upon circumscribed circles.The proof starts with the statement of the following theorem,

Theorem 3. *Let γ be a smooth simple closed curve in the plane. Denote by C the circumscribed circle about γ. Then*

    1. $\gamma \cap C$ contains at least two points;

    2. if $\gamma \cap C$ has at least n components and $n > 1$, then γ has at least 2n vertices.

He proves this theorem using four lemmas. His first lemma is

Lemma 1. *Let E be a compact set in the plane containing at least two points. Then among all circles with the property that the closed disk bounded by C includes E, there is a unique one of minimum radius $R > 0$.*

This circle that is described in the preceding lemma is called the circumscribed circle about E. This leads to the second lemma,

Lemma 2. *If C is the circumscribed circle about E, then any arc of C greater than a semicircle must intersect E.*

Both of these lemmas come from the fact that if you assumed the opposite then you could find a smaller circle enclosing E. The reasoning behind this can, partly, be seen in Figure 4.

Osserman then states one more lemma that he uses to prove his first theorem. The last lemma is

Lemma 3. *Let a smooth oriented curve γ have the same unit tangent at a point P as a positively oriented circle C of radius R. Let k be the curvature of γ. Then if $k(P) > 1/R$, a neighborhood of P on γ lies inside C, while if $k(P) < 1/R$, a neighborhood of P on γ lies outside C.*

Figure 4: An example showing that if the circle only intersects the curve at one location, then it cannot be the circumscribed circle.

What this is really saying is that if you have a general curve and a circle that meet at a point, P, have the same orientation at that point, and have the same tangent vector, $\vec{t}$, and therefore the same normal, $\vec{n}_s$, at that point, then if the curvature of the curve is greater than the curvature of the circle then a portion of the curve around that point is inside of the circle and if the curvature of the curve is less than that of the circle then a portion of the curve around that point is outside of the circle. A basic example of this theorem can be seen in Figure 5. Since the curvature of a circle of radius R is given by $k = \frac{1}{R}$ it is clear that since the outer circle has the larger radius, it also has the smaller curvature.

A simple argument for this idea is as follows, consider the Taylor polynomials for both the circle and the curve. For the circle the second order Taylor polynomial would be given by

$$P + s\vec{t} + \frac{s^2}{2}\frac{1}{R}\vec{n}_s$$

and the Taylor polynomial for the curve would be given by

$$P + s\vec{t} + \frac{s^2}{2}k(P)\vec{n}_s.$$

Clearly the only difference between these two equations is that the curvature of the circle is known to be $\frac{1}{R}$ while the curvature of the curve is given by $k(P)$. This means that the curvature alone will determine the difference in the local behavior of the curve when
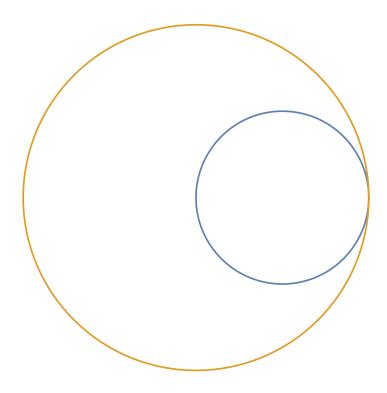
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compared to the circle.

Figure 5: An example of two circles that meet at one point.

In order to prove the theorem, Osserman first uses the following lemma

Lemma 4. *Let γ be a positively oriented Jordan curve, C the circumscribed circle and P_1, P_2 points of $\gamma \cap C$. Let γ_1 be the (positively oriented) arc of γ from P_1 to P_2. Then either γ_1 coincides with the circular arc $P_1 P_2$ or else there is a point Q_1 on C satisfying $k(Q_1) < \frac{1}{R}$ where R is the radius of C.*

Note that if γ_1 coincides with the circular arc $P_1 P_2$. Then, because of this on either side of the overlapping section there must be two points of the curve where the curvature is greater than that of the circle. Otherwise, the curve would be unable to separate from the circle.

The proof of this lemma is as follows. Given a curve γ and its circumscribed circle C. Now consider an arc along γ from two points P_1 and P_2. Now by Lemma 2 we can assume

that P_1 and P_2 are separated by no more than a semi-circle because if they are we could just pick a point between them and consider the arc formed by that point and either P_1 or P_2. Now to prove this lemma, we look at the case where the arc $P_1 P_2$ does not coincide with the circle. This means there is at least one point Q inside of C. In order to visualize this proof we assume that the circle is arranged such that the line connecting P_1 and P_2 is a vertical line and that the point Q is located to the right of the line. We can then form a new circle using the points P_1, P_2, and Q. One important point to note is that since Q is strictly inside of C, and since $P_1 P_2$ is at most a semicircle, the radius of this new circle is strictly greater than the radius of C. This means that the curvature of the new circle is strictly less than the curvature of C since the curvature of a circle is given by $\frac{1}{R}$. This new circle can the be moved horizontally away from C, and to the left, until the last point, say Q_1 where it coincides with γ. Now the entire arc, except for Q_1, of γ from P_1 to P_2 lies outside of this new circle. If we apply Lemma 3 to this point and using the fact that γ is outside of the circle, we conclude that the curvature of Q_1 is less than or equal to the curvature of the new circle. Since the curvature of Q_1 is less than the curvature of the new circle which is strictly less than the curvature of C, we can conclude that $k(Q_1) < \frac{1}{R}$.

Using this lemma the proof of the theorem is now relatively straight forward. Consider a curve γ with a circumscribed circle C. The set $\gamma \cap C$ will have $n > 1$ components and we pick a point in each. This will create n arcs of γ not coinciding with C between the intersection points. If we consider the endpoints of these arcs and use Lemma 3 we can conclude that the curvature of the endpoints is greater than or equal to the curvature of C, since γ lies inside of C. Also, if we use Lemma 4 we can obtain a point within each arc where the curvature is strictly less than that of C. So we have n arcs and each arc has an endpoint, where the curvature is greater than that of C and an interior point where the curvature is less than that of C. This means that we know the curvature has to fluctuate up and down at least n times, meaning that it must have at least n maxima and at least n minima. This means that it must then have at least $2n$ vertices.

So, for the case where $n > 1$ we are done because we then have at least four vertices. For the special case of $n = 1$ this is slightly more complicated. In this case the curve would have to coincide with the circle for an arc larger than a semicircle. But, we know that there must be some section where the curve does not overlap the circle, so if we apply the same logic as we did to the overlapping case before, we can conclude that there must be a point on either side of the overlap where the curvature is greater than that of the circle. Also, by using Lemma 4, we can conclude that there has to be one point in the non-overlapping section where the curvature of that point is less than the curvature of the circle. So then as can be seen in Figure 6, this forces there to be 4 vertices in the curvature.
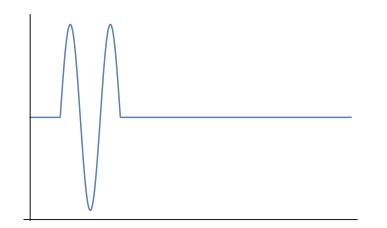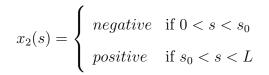

Figure 6: An example of a possible curvature graph for the case where $n = 1$

2.3   Herglotz's Proof

Another interesting proof is the one published by G. Herglotz in 1930 (see [2]). His proof is relatively straightforward, but it is only for the case of convex curves and it is a proof by contradiction.

The proof starts by assuming that there is a curve C that has only two vertices, which we call M and N. We can then assume that the line connecting M and N does not cross C at any other point because then C would not be convex. Then we can define an axis x_1, that is the line connecting M and N and define $\gamma(s) = (x_1(s), x_2(s))$ where s is the arc length so

that $(x_1(0), x_2(0)) = M$ and $(x_1(s_0), x_2(s_0)) = N$. Note that then we can say

$$x_2(s) = \begin{cases} negative & \text{if } 0 < s < s_0 \\ positive & \text{if } s_0 < s < L \end{cases}$$
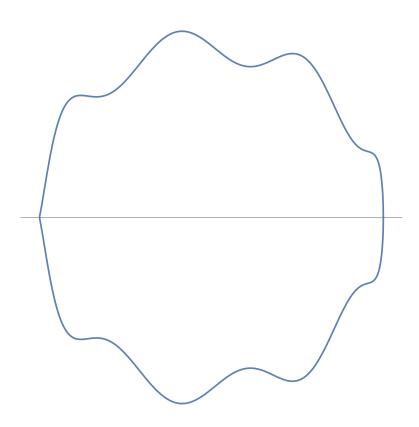

Figure 7: An example of what $x_2(s)$ could look like. The points where it crosses the axis are at s_0 and 0/L

Here L is the length of the curve, C. From Figure 7 it is clear that the curve wraps back around on itself, so that the points 0 and L are really the same point and have the same value, which is 0. The tangent vector would then be given by, $\mathbf{t} = (\dot{x}_1, \dot{x}_2)$ and $\mathbf{n_s} = (-\dot{x}_2, \dot{x}_1)$ as defined by equations (1) and (2) respectively.

So then using equation (3), we get the equations

$$\ddot{x}_1 = -k_s \dot{x}_2$$

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and

$$\ddot{x}_2 = k_s \dot{x}_1$$

From this, it is clear that

$$\int_0^L k_s \dot{x}_2 ds = -\int_0^L \ddot{x}_1 ds = -\dot{x}_1 \big|_0^L = 0$$

We can then rewrite this as

$$0 = \int_0^L k_s \dot{x}_2 ds = \int_0^{s_0} k_s \dot{x}_2 ds + \int_{s_0}^L k_s \dot{x}_2 ds$$

Now, recall the second mean value theorem. This says that if you have two functions $f(x)$ and $g(x)$ where $f(x)$ is monotone then the following relation holds

$$\int_a^b f(x)g(x)dx = f(a) \int_a^e g(x)dx + f(b) \int_e^b g(x)dx$$

for some $e \in (a, b)$.

Using this fact we can write

$$\int_0^{s_0} k_s \dot{x}_2 ds = k_s(0) \int_0^{e_1} \dot{x}_2 ds + k_s(s_0) \int_{e_1}^{s_0} \dot{x}_2 ds$$

and

$$\int_{s_0}^L k_s \dot{x}_2 ds = k_s(s_0) \int_{s_0}^{e_2} \dot{x}_2 ds + k_s(L) \int_{e_2}^L \dot{x}_2 ds$$

Then evaluating the integrals gives us

$$\int_0^{s_0} k_s \dot{x}_2 ds = k_s(0) \left[x_2(e_1) - x_2(0) \right] + k_s(s_0) \left[x_2(s_0) - x_2(e_1) \right]$$

and

$$\int_{s_0}^L k_s \dot{x}_2 ds = k_s(s_0)[x_2(e_2) - x_2(s_0)] + k_s(L)[x_2(L) - x_2(e_2)]$$

Now we know, from the way that we defined x_2 that $x_2(0) = x_2(L) = x_2(s_0) = 0$, so what we are left with is

$$\int_0^{s_0} k_s \dot{x}_2 ds = k_s(0)x_2(e_1) - k_s(s_0)x_2(e_1)$$

and

$$\int_{s_0}^{L} k_s \dot{x}_2 ds = k_s(s_0)x_2(e_2) - k_s(L)x_2(e_2)$$

Putting these equations back together, and noting that $k_s(0) = k_s(L)$ we have

$$\int_0^{L} k_s \dot{x}_2 ds = k_s(0)x_2(e_1) - k_s(s_0)x_2(e_1) + k_s(s_0)x_2(e_2) - k_s(0)x_2(e_2)$$

$$= [k_s(0) - k_s(s_0)][x_2(e_1) - x_2(e_2)]$$

We also know from above that this integral is equal to 0, so this implies that

$$k_s(0) - k_s(s_0) = 0 \text{ or } x_2(e_1) - x_2(e_2) = 0.$$

However, $k_s(0) - k_s(s_0)$ cannot be 0 since one is the maximum and the other is the minimum of the curvature and $x_2(e_1) - x_2(e_2)$ cannot be zero because $x_2(e_1)$ is defined to be negative and $x_2(e_2)$ is defined to be positive, meaning the whole term $x_2(e_1) - x_2(e_2)$ must be negative.

This is a contradiction, meaning that the curve must have at least one more vertex, and since vertices occur in pairs, there must be at least two more.

2.4 Discussion

Many interesting aspects can be considered in relation to the four vertex theorem. The following result is from Jackson [3].

Proposition 4. *At every interior point (or circular arc) of a monotone arc, A, the arc crosses its circle of curvature. The crossing is from right to left or from left to right according*

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as the curvature is monotone nondecreasing or monotone nonincreasing.

A monotone arc means that curvature function of the arc is monotone. This corollary is best made sense of through an example.

Example 3. Consider the case of the spiral, which has either strictly increasing or decreasing curvature. In this case, as can be seen in Figure 8, we consider the spiral defined by $\gamma(t) = (-t\cos(t), -t\sin(t))$. We then calculate the curvature to be

$$k(t) = \frac{t^2 + 2}{(1 + t^2)^{3/2}}.$$

So then if we pick a point on the spiral and plot its circle of curvature, we can clearly see that the spiral immediately crosses it.
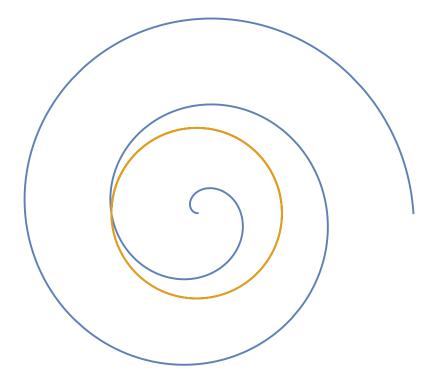

Figure 8: A pictorial explanation of Corollary (4).

3 The Converse of the Four Vertex Theorem

The full statement of the converse is as follows,

Theorem 4. *Any continuous real-valued function on the circle that has at least two local maxima and two local minima is the curvature of a simple closed curve in the plane.*

We will look at a proof of a modified version of the converse which is only for the case of strictly positive curvature. This proof is based upon the work by Herman Gluck in 1971. The full converse was not proved until 1997 by Björn Dahlberg.

Gluck's proof is based upon a winding number argument in the group of diffeomorphisms of the circle. The idea is to start with any continuous strictly positive curvature $k : S^1 \to \mathbb{R}$ and then we can think of the parameter along S^1 as the angle of inclination, θ, of the desired curve, where $\theta = \phi(s)$ in equation (5). Since k is an arbitrary curvature there is no guarantee that it will close up, but we can construct another curve that will. This means that we identify S^1 with the parameter space $[0, 2\pi]$. So then there is a unique map $\alpha : [0, 2\pi] \to \mathbb{R}^2$ that starts at the origin and has a unit tangent vector $(\cos(\theta), \sin(\theta))$ and has curvature given by $k(\theta)$ at $\alpha(\theta)$. So then we can think of $E = \alpha(2\pi) - \alpha(0)$ as the error vector of α and say that it measures the curves failure to close up.

Next we find a loop of diffeomorphism, h, of the circle that is contractible in the group $Diff(S^1)$. So we then construct $\alpha_h : [0, 2\pi] - \mathbb{R}^2$ as above except the curvature at the point $\alpha(\theta)$ is equal to $k \circ h(\theta)$ instead. The main point of this proof is guaranteeing that one of these, $\alpha_h(\theta)$, will close up. Since h is a diffeomorphism, we can guarantee that it has an inverse, h^{-1}. So then if we look at $\alpha_h(\theta) = \alpha_h \circ h^{-1} \circ h(\theta)$ and let $h(\theta) = t$ then the curvature at the point $\alpha_h \circ h^{-1}(t)$ is k(t). Therefore $\alpha = \alpha_h \circ h^{-1}$ is a reparametrization that closes up and has curvature $k(t)$.
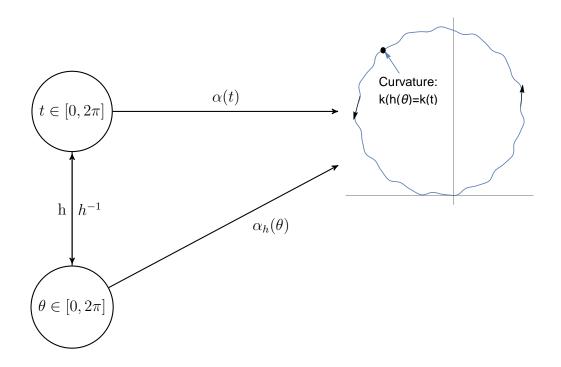
Figure 9: A diagram of the different ways of viewing the relationship between the different functions and parameter spaces used in Gluck's proof.
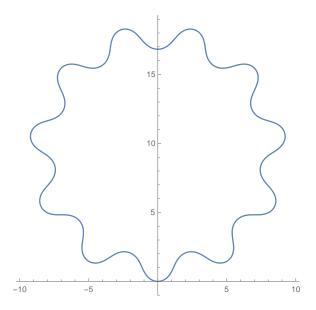

Figure 10: An example of a function that was graphed based upon it's curvature function, in this case $k(s) = \frac{1}{11} + \cos(s)$.

We will consider some examples of curves derived from there curvature based upon work by Arroyo, Garay, and Mencía [4]. In their paper Arroyo et.al. give prescriptions for how to

determine when a periodic curvature function is the curvature of a curve that closes up and also the period of said curve. Using their prescription and equations (6) and (7) I created Figures 10 and 11.
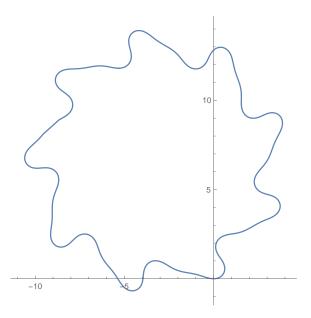

Figure 11: An example of a function that was graphed based upon it's curvature function, in this case $k(s) = \frac{1}{9} + \cos(s) + \sin(2s)$.

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