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The Axiom of Choice in Topology

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Abstract

Cantor believed that properties holding for finite sets might also hold for infinite sets. One such property involves choices; the Axiom of Choice states that we can always form a set by choosing one element from each set in a collection of pairwise disjoint non-empty sets. Since its introduction in 1904, this seemingly simple statement has been somewhat controversial because it is magically powerful in mathematics in general and topology in particular.

In this paper, we will discuss some essential concepts in topology such as compactness and continuity, how special topologies such as the product topology and compactification are defined, and we will introduce machinery such as filters and ultrafilters. Most importantly, we will see how the Axiom of Choice impacts topology.

Most significantly, the Axiom of choice in set theory is the foundation on which rests Tychonoff's Infinite Product Theorem, which people were stuck on before the axiom of choice was applied. Tychonoff's Theorem asserts that the product of any collection of compact topological spaces is compact. We will present proofs showing that the Axiom of Choice is, in fact, equivalent to Tychonoff's Theorem. The reverse direction of this proof was first presented by Kelley in 1950; however, it was slightly flawed. We will go over Kelley's initial proof and we will give the correction to his proof. Also, we introduce the Boolean Prime Ideal Theorem (a weaker version of the Axiom of Choice), which is equivalent to Tychonoff's Theorem for Hausdorff spaces. Finally, we will look at an interesting topological consequences of the Axiom of Choice: the Stone-Cech Compactification. We will see how the Stone-Cech Compactification is constructed from ultrafilters, whose existence depends on the Axiom of Choice.

1 Zermelo-Fraenkel Set Theory

A hundred or so years ago, mathematicians were interested in the foundations of mathematics. Namely, they wanted to know what constitued a suitable foundation. A standard idea was to base mathematics on the notion of a set, which is a collection of distinct objects.

Georg Cantor began, and Gottlob Frege continued, the study of a version of set theory called Naive set theory. It is a naive theory in the sense that "naive set theory" is a non-formalized theory; that is, a theory that uses a natural language to describe sets and operations on sets. Unfortunately, Bertrand Russell pointed out a major problem with naive set theory: it's inconsistent! This was Russell's paradox.

Russell's Paradox Naive set theory declares that any definable collection is a set. However, Bertrand Russell in 1901 pointed out that some attempted formalizations of naive set theory lead to a contradiction. Let $R = \{x : x \notin x\}$. Then $R \in R \longleftrightarrow R \notin R$, contradiction.

So does Russell's paradox mean that mathematics might be inconsistent? Eventually Zermelo and Fraenkel worked out an alternative set of axioms, now called the Zermelo-Fraenkel Axioms (ZF), which could serve as a foundation of mathematics but that did not appear to be inconsistent. Unrestricted comprehension gets replaced by restricted comprehension (Axiom schema of specification), which says that if you have a property and a set then you can write down the subset of that set of elements which satisfy that property.

In many cases a sequence of selections (the construction of a choice function) can be made without invoking the axiom of choice. In particular, this is true if the number of sets is finite, or if a selection rule is available: a distinguishing property that happens to hold for exactly one object in each set. For instance, for any collection of pairs of shoes, we can simply select the left shoe from each pair, and thus form an appropriate choice function. Also, if we have a collection of sets S_i of the form $\{a, b\}$ where $a, b \in R$, we can pick one element from each set by defining $f(S_i) = min\{a, b\}$. However, these are very special cases in which the nature of the individual non-empty sets makes it possible to avoid the use of the Axiom of choice.

Choice Function A choice function is a mathematical function f that is defined on some collection X of nonempty sets and assigns to each set S in that collection some element $f(S)$ of S.

Now recall how we define the cartesian product of two non-empty sets X and Y. The cartesian product is

$$X \times Y = \{(x, y) : x \in X, y \in Y\}.$$

Using this same idea, we can define the cartesian product of any finite collection of non-empty sets. Suppose S_i with $i = 1, ..., n$ is a finite collection of non-empty sets. Then

$$S_1 \times ... \times S_n = \{(s_1, ..., s_n) : s_1 \in S_1, ..., s_n \in S_n\}.$$

However, the same rule fails when we look at the cartesian product of an uncountable collection of sets. Instead of enumerating the sets, we can define the cartesian product of an arbitrary (possibly infinite) indexed family of sets.

Definition 1.1. Cartesian Product (uncountable collection of sets)

Let I be any index set and S_i with $i \in I$ be a collection of sets. Then the *cartesian product* of the collection S_i is:

$$\Pi S_i = \{f : I \to \cup S_i : f(i) \in S_i \; for \; all \; i \in I\}.$$

Also, for each $j \in I$, the j^{th} projection map $\pi_j : \Pi S_{i \in I} \to S_j$ is defined by $\pi_j(f) = f(j)$.

1.1 Finite Axiom of Choice

One weaker version of the Axiom of Choice is the Finite Axiom of Choice. We can prove this theorem from ZF and the usual rules of inference.

Theorem 1.2. *(ZF)*

If S is any finite collection of nonempty sets, then there exists a choice function on S.

This finite Axiom of Choice is the "weakest" version because it can be proved directly from rules of inference, without referring to the Axiom of Choice. We will be making use of the following rule of inference:

Definition 1.3. Existential Instantiation: $\exists x \phi(x) \to \phi(c)$

We start by looking at a collection A consisting of one non-empty set X. Since X is non-empty, $\exists\omega(\omega \in X)$. Thus we can conclude $c \in X$. Let our choice function $f = \{(X, c)\}$ and we are done.

Now we look at the finite collection of non-empty sets and let $S = \{S_1, ..., S_n\}$. It follows that $\forall S_i(\exists\omega_i \in S_i)$. By our rule of inference, we have

$$From\ \exists x \in S_1,\ we\ can\ conclude\ c_1 \in S_1.$$

$$...$$

$$From\ \exists x \in S_n,\ we\ can\ conclude\ c_n \in S_n.$$

Finally, our choice function $f = \{(S_1, c_1), ..., (S_n, c_n)\}$.

2 The Axiom of Choice

The Axiom of Choice is no doubt one of the most powerful as well as controversial axioms in mathematics. The biggest problem with the Axiom of Choice is that it yields the existence of some objects that are not definable or cannot be explicitly constructed. So naturally arguments against the use of this axiom arose. However, despite this paradoxical debate, the Axiom of Choice is still implicitly and widely used because it allows us to prove many significant results such as Tychonoff's Theorem. Nowadays, this axiom is accepted by most mathematicians and is included in the standard form of axiomatic set theory.

First, let us see what the Axiom of Choice says:

For every family F of nonempty sets, there is a choice function f such that $f(S) \in S$ for each set S in the family F.

In other words, the Axiom of Choice states that the cartesian product of any collection of non-empty sets is not empty.

Informally speaking, the Axiom of Choice allows us to choose exactly one element from each set in a collection of non-empty sets. Let's look at an example to make it more clear. Let F be the collection of singleton sets. We can select the singleton from each set, which forms a proper choice function. In this example, the existence of a choice function is unquestionable. However, the Axiom of Choice is more functional when there is more than one element in each set in the collection. The Axiom of Choice guarantees the existence of a choice function although we cannot explicitly construct one. In fact, the Axiom of Choice allows us to prove many "obviously true" results. Nonetheless, we fail to validate the Axiom of Choice from ZF.

While the Axiom of Choice is undoubtably powerful in mathematics, we should not dismiss the power of its weaker equivalents. Although the idea of choice underlies many theorems, in some cases the Axiom of Choice is not fully required. Instead, its weaker equivalents suffice to prove many results. Let us look at some widely used weaker equivalents of the Axiom of Choice:

In mathematics, the axiom of dependent choice, denoted DC, is another weak form of the Axiom of Choice that is still sufficient to develop most of real analysis. It was introduced by Bernays (1942).

Theorem 2.1. *Axiom of Dependent Choice (DC): For any nonempty set X and any entire binary relation R on X, there is a sequence* (x_n) *in X such that* $x_n\ R\ x_{n+1}$ *(where R is any binary relation) for each n in N.*

We can prove DC from AC as follow:

Let R be a binary relation on a non-empty set S such that $\forall a \in S$, $\exists b \in S$ such that $a\ R\ b$. For all $a \in S$, let R(a)= $\{b \in S : a\ R\ b\}$. By our assumption, R(a) is not empty for all $a \in S$, so that $\{R(a)\}_{a\in S}$ is an indexed collection of non-empty sets.

Using the Axiom of Choice, there exists a choice function $f : S \to S$ such that: $\forall a \in S$, $f(a) \in R(a)$, from which follows $a\ R\ f(a)$.

So for any $x \in S$, the sequence: $< x_n >_{n\in N} = < f^n(x) >_{n\in N}$ where f^n denotes the composition of f with itself n times, is a sequence such that $x_n\ R\ x_{n+1}$.

On the other hand, it is known that the Axiom of Dependent Choice does not imply the Axiom of Choice.

Theorem 2.2. *Axiom of Countable Choice (CC):*

Any countable collection of non-empty sets must have a choice function.

Paul Cohen showed that the Axiom of Countable Choice is not provable in Zermelo-Fraenkel set theory (ZF) without the axiom of choice. Moreover, CC is strictly weaker than DC and is provable from DC alone:

Proof. :

Given non-empty sets S_n for $n = 0, 2, ...$, we know that for all n, $S_0 \times ... \times S_n$ is not empty by the Axiom of finite choice.

Let $S = \cup_{n \in N} S_1 \times ... \times S_n$ and let R be the binary relation on S such that

$$(a_0, ..., a_n, b)\, R\, (c_0, ..., c_m, d) \iff n = m\, and\, a_i = c_i\, for\, i = 0, ..., n\, and\, b, d \in S_{n+1}.$$

DC yields $(a_0)\, R\, (a_0, a_1)\, R\, (a_0, a_1, a_2)\, R ...$, which follows that $\{a_0, a_1, ...\}$ is in the choice set.

□

Given the axioms of Zermelo-Fraenkel Set Theory, the Axiom of Dependent Choice is insufficient to prove that there is a non-measurable set of reals, or that there is a set of reals without the property of Baire or without the perfect set property. In other word, the Axiom of Dependent Choice is only sufficient to prove several "good" properties in Topology.

On the other hand, many equivalents to the Axiom of Choice are as powerful. In this thesis, we will use the following direct equivalent to the Axiom of Choice as an important tool for many proofs:

Lemma 2.3. *Zorn's Lemma*

Suppose a partially ordered set P has the property that every chain has an upper bound in P. Then the set P contains at least one maximal element.

3 General Topology (Point-Set Topology)

Given that our focus is the application of the Axiom of Choice in topology, we will implicitly introduce topology by addressing several questions:

Which field of study is topology generated from? What are the most important concepts in topology? How is topology related to the Axiom of Choice?

3.1 Introduction to General Topology

The concept of topology originated in the 17th century, where Gottfried Leibniz envisioned the geometria situs (Greek-Latin for "geometry of place") and analysis situs (Greek-Latin for "picking apart of place"). The term topology was introduced by Johann Benedict Listing in the 19th century, although it was not until the first decades of the 20th century that the idea of a topological space was developed. By the middle of the 20th century, topology had become a major branch of mathematics.

Modern topology depends strongly on the ideas of set theory, and is concerned with the properties of space that are preserved under continuous deformations, such as stretching and bending, but not tearing or gluing. This can be studied by considering a collection of subsets, called open sets, that satisfy certain properties, turning the given set into what is known as a topological space.

Topology is a big topic and it has has many subfields. In this thesis, we are focusing on properties of general topology, which is also called point-set topology. General topology establishes some foundational aspects of topology and investigates properties of topological spaces and concepts inherent to topological spaces. Fundamental concepts such as continuity, compactness, and connectedness are used in all other branches of topology. Let us start this section by looking at some basic concepts in topology.

Definition 3.1. A topology on a set X is a collection τ of subsets of X, called the open sets, satisfying:

(1) Any union of elements of τ belongs to τ.

(2) Any finite intersection of elements of τ belongs to τ.

(3) $\emptyset$ and X belong to τ.

We say (X, τ) is a topological space, sometimes abbreviated "X is a topological space" when no confusion can result about τ. Every element we add to the topology is defined to be an open set.

The "nicest" topological spaces always allow distinct points to be separated by open sets. These are called Hausdorff spaces:

Definition 3.2. A space X is a Hausdorff space if and only if whenever x and y are distinct points of X, there are disjoint open sets U and V in X with $x \in U$ and $y \in V$.

Examples of Topological spaces

1. The Usual Topology on R

   The Usual Topology on R can be expressed as the collection of unions of open intervals. For example, (a, b) is in the topology on R, as is $(1, 3) \cup (5, 9)$.

2. The Usual Topology on R^2

   The Usual Topology on R^2 can be expressed as the collection of unions of open discs.

3. Metric Topology

   Definition 3.3. A metric space is an ordered pair (M, ρ) consisting of a set M together with a function $\rho : M \times M \to R$ satisfying for $x, y, z \in M$:

   (1) $\rho(x, y) \geq 0$.

   (2) $\rho(x, x) = 0$ and $\rho(x, y) = 0$ implies $x = y$.

   (3) $\rho(x, y) = \rho(y, x)$.

   (4) $\rho(x, y) + \rho(y, z) \geq \rho(x, z)$.

   Let (M, ρ) be a metric space. Then the collection of open sets in M (a set A is open in M if for each $a \in A$ there is an open disc about a contained in A) form a topology on M, which is the metric topology.

4. The Discrete Topology

   Let X be any set and let τ be the collection of all subsets of X. Then τ is clearly a topology for X. Further, this topology is called the discrete topology.

5. Trivial Topology

   Let X be any set and let $\tau = \{\emptyset, X\}$. Then τ is called the trivial topology for X.

Close Set If X is a topological space and $E \subset X$, we say E is closed if and only if $X - E$ is open.

We can also have "clopen sets" in a topology, referring to sets that are both open and closed. A set can be both.

Theorem 3.4. *If F is the collection of closed sets in a topological space X, then*

(1) Any intersection of members of F belongs to F.
(2) Any finite union of members of F belongs to F .
(3) X and $\emptyset$ both belong to F.

3.2 Related Definitions

As in set theory, the study of topology requires basic concepts and definitions. For instance, a point in space can have a "neighborhood" and a topology can be generated from "bases" and "subbases."

Definition 3.5. Neighborhood

If X is a topological space and $x \in X$, a neighborhood of x is a set U which contains an open set V containing x. The collection U_x of all neighborhoods of x is the neighborhood system at x.

Definition 3.6. Neighborhood Base

A neighborhood base at x in the topological space X is a subcollection B_x taken from the neighborhood system U_x, having the property that each $U \in U_x$ contains some $V \in B_x$. That is,

$$U_x = \{U \subset X : \ V \subset U \text{ for some } V \in B_x\}.$$

Examples

1. In any topological space, the open neighborhoods of x form a neighborhood base at x.

2. In the trivial topology on X, the only neighborhood base at $x \in X$ is the collection containing the single set X.

Definition 3.7. Finite Intersection Property

Let X be a set and A a collection of subset of X. The collection A is said to have the finite intersection property (FIP) if the intersection of any finite subcollection of A is nonempty.

3.2.1 Decomposition of Topologies

Definition 3.8. Base

If (X, τ) is a topological space, a base for τ is a collection $B \subset \tau$ such that every set in τ is a union of some sets in B.

In other words, τ can be recovered from B by taking all possible unions of subcollections from B. For example, the collection of all open intervals is a base for the usual topology (where every open interval is an open set) on R.

Definition 3.9. Subbase

If (X, τ) is a topological space, a subbase for τ is a collection $L \subset \tau$ such that the collection of all finite intersections of elements from L forms a base for τ.

For example, the collection of sets of the form $(-\infty, a)$ and (b, ∞) is a subbase for the usual topology on real line.

3.3 Highlighted Properties

In this section, we discuss some fundamental concepts in point-set topology: continuity, compactness, and connectedness. These concepts are similar to what we learned in set theory but slightly different. For instance, informally speaking, in the context of general topology, the idea of "arbitrarily small" is replaced by "well-chosen open sets." Let's take a closer look at these properties.

We all have seen continuous functions in set theory before. In topology, a continuous function is one of the essential concepts. Now let us see the definition of continuous function in topology:

Definition 3.10. A function $f : X \to Y$ between two topological spaces X and Y is continuous if for every open set $V \subseteq Y$, the inverse image $f^{-1}(V) = \{x \in X : f(x) \in V\}$ is an open subset of X.

The following theorem introduces the properties of continuous function in topology.

Theorem 3.11. *Let X and Y be two topological spaces and $f : X \to Y$. Then the following statements are equivalents:*

(1) The function f is continuous.

(2) For each open set $H \in Y$, $f^{-1}(H)$ is open in X.

(3) For each closed set $H \in Y$, $f^{-1}(H)$ is closed in X.

Examples

1. Let X be a topological space. Let $f : X \to X$ be the identity function ($f(x) = x$), then $f^{-1}(U) = U$ for any open set $U \subseteq X$. Thus the identity function is continuous.

2. If $X = \Pi_{i \in I} X_i$, then the canonical projections $\pi_i : X \to X_i$ are continuous according to the definition of product topology. The proof of this is straightforward:

   Suppose $X = \Pi_{i \in I} X_i$ is the product topology equipped with projection mappings. We will show that some arbitrary projection $\pi_j : X \to X_j$ is continuous. Let U be an open set in X_j. But then $\pi_j^{-1}(U)$ is defined to be a basic open set in the product topology, as desired.

Compactness Informally speaking, compact sets are those that can be covered by finitely many sets of arbitrarily small size.

Definition 3.12. Let A be a subset of the topological space X. An open cover for A is a collection O of open sets whose union contains A. A subcover derived from the open cover O is a subcollection S of O whose union contains A.

Definition 3.13. A topological space X is compact provided that every open cover of X has a finite subcover.

Examples

1. Any finite topological space is compact. We can use the Finite Axiom of Choice to select exactly one open set for each element in the space. This forms a finite subcover.

2. Take any set X, and define the cofinite topology on X by declaring a subset of X to be open if and only if it is empty or its complement is finite. Then X is a compact topological space. For any given cover, we first select one open set in the cover. Then only finitely many element are not covered by this subset because by definition, all open sets in the topology are cofinite. So we can pick one corresponding set for each uncovered element and add these sets to our previous subset, which forms a finite subcover of X.

Definition 3.14. A topological space X is said to be disconnected if and only if there are two disjoint nonempty open sets H and K in X such that $X = H \cup K$. Otherwise, X is said to be connected.

For example, any discrete space of more than one point is disconnected. We take a look at a two-point discrete space. Let A be any proper subset of X. Then both A and A^c are non-empty are open subsets of X such that

$A \cup A^c = \emptyset$ and $A \cup A^c = X$. This shows that $\{A, A^c\}$ is a disconnection of X, so X is a disconnected space.

4 Filters and Ultrafilters

Filters and ultrafilters play an important role in topology. For example, they allow for a general theory of convergence in topological spaces. A filter defines a notion of largeness for subsets of S: the large sets are the members of S, and the small sets are the sets whose complements are large. On the other hand, an ultrafilter is a maximal filter, demanding that every set be either large or small.

4.1 Introduction to Filters and Ultrafilters

Definition 4.1. A filter on a set X is a collection F of subsets of X satisfying:

1. If $F_1, F_2 \in F$, then $F_1 \cap F_2 \in F$.
2. If $F' \in F$ and $F' \subset F''$, then $F'' \in F$.

For example, let $a \in S$ be arbitrary. The family $\{A \subseteq S : a \in A\}$ is called the principal filter generated by a.

Definition 4.2. A filter F on X is an ultrafilter if and only if for all $E \subset X$, either $E \in F$ or $(X - E) \in F$.

By definition, principal filters are ultrafilters, but the trivial filter and the cofinite filter are not.

Notation If X is a topological space and $x \in X$, let $u_x = \{u \subseteq X : u \textit{ is open in } X \textit{ and } x \in u\}$.

Definition 4.3. A filter F on a topological space X is said to converge to $x \in X$ (written $F \to x$) if $u_x \subset F$.

Theorem 4.4. *Ultrafilter Convergence Theorem*

A topological space is compact if and only if every ultrafilter in X converges to at least one point.

Proof. We proceed by contradiction. Let F be an ultrafilter on Y and suppose Y is compact but F converges to no point. Then for all $y \in Y$ there exists some U_y (which is a neighborhood of y) with $U_y \notin F$. It follows that

$\{U_y : y \in Y\}$ is an open cover of Y. Since Y is compact by assumption, this open cover has a finite subcover. But $Y \in F$ so that some neighborhood of y is in F as well, contradiction.

Conversely, suppose Y is not compact. Let $\{U_i : i \in I\}$ be an open cover on Y with no finite subcover. It follows that $\cap_i (Y - U_i) = \emptyset$ but $\cap_i (Y - U_i)$ has the finite intersection property. Let U be an ultrafilter generated from $\{(Y - U_i) : i \in I\}$. For any point $y \in Y$, we can choose corresponding U_i such that $y \in U_i$ but $U_i \notin F$ (because F is ultrafilter and $Y - U_i \in F$), so F does not converge to any $y \in Y$.

□

4.2 Boolean Prime Ideal Theorem

The Boolean Prime Ideal Theorem is another strictly weaker version of the Axiom of Choice. In this section, we will introduce how this theorem is applied in topology. To begin with, we go over several useful definitions, theorems and lemmas.

Definition 4.5. Ideal

A non-empty subset I of a partially ordered set (P, $\leq$) is an ideal if the following conditions hold:

1. For every $x \in I$, $y \leq x$ implies that $y \in I$.

2. For every x, y in I, there is some element $z \in I$ such that $x \leq z$ and $y \leq z$.

Definition 4.6. Filter on Partially Ordered Set

A filter on a partially ordered set is a non-empty set such that any two elements in F have a common lower bound and all elements above an element in F are also in F.

Definition 4.7. Ultrafilter on Partially Ordered Set

An ultrafilter on a given partially ordered set P is a maximal filter on P, that is, a filter on P that cannot be enlarged.

Definition 4.8. Prime Ideal

An ideal I is a prime ideal on S if and only if the complement of I is an ultrafilter.

Definition 4.9. In a partially ordered set P, the join and meet of a subset S are respectively the least upper bound of S and the greatest lower bound of S.

Definition 4.10. A lattice is a partially ordered set in which every two elements have a unique least upper bound and a unique greatest lower bound.

Definition 4.11. A distributive lattice is a lattice in which the operations of join and meet distribute over each other.

Theorem 4.12. *Every Maximal Ideal is prime*

Let L be a distributive lattice. Let F be a filter in L and M be an ideal in L which is disjoint from F. If no ideal in L larger than M is disjoint from F, then M is a prime ideal.

Next let us see the statement of the Boolean Prime Ideal Theorem:

Theorem 4.13. *Boolean Prime Ideal Theorem (stronger version)*

Let B be a Boolean algebra, let I be an ideal and let F be a filter in B, such that I and F are disjoint. Then I is contained in some prime ideal of B that is disjoint from F.

Now we look at a proof of the Boolean Prime Ideal Theorem from the Axiom of Choice.

Theorem 4.14. *The Axiom of Choice implies the Boolean Prime Ideal Theorem.*

Proof. Let (S, $\leq$) be a boolean algebra. Let I be an ideal in S and let F be a filter on S disjoint from I. Let T be the set of ideals in S that contain I and are disjoint from F, partially ordered by inclusion. We pick an arbitrary chain N in T, in the form $N_1 \subseteq ... \subseteq N_n \subseteq ...$. First, we want to show that the chain N has an upper bound. Let U= $\cup N_i$ be the union of all elements in N. Since every element in N is disjoint from F and contains I, we thus know that U is disjoint from F and that U contains I. This collection U is an upperbound for the chain N just because it is the union of all elements in N. Next we will show that $U \in T$; that is U is also an ideal on S.

First pick an arbitrary $x \in U$ and an arbitrary y with $y \leq x$. It follows that $\exists N_k \in N$ such that $x \in N_k$. Since N_k is an ideal, by definition we have $y \in N_k$. Hence, $y \in U$

Secondly pick $x, y \in U$. It follows that $\exists N_i, N_j \in N$ such that $x \in N_i$ and $y \in N_j$. Because N_i, N_j are in the totally ordered chain N, we know that either $N_i \subseteq N_j$ or $N_j \subseteq N_i$. Suppose without loss of generality that $N_i \subseteq N_j$. Then $x, y \in N_j$. By our assumption, N_j is an ideal, thus $\exists z \in N_j$ with $x \leq z$ and $y \leq z$. Further, since U is the union of all elements in N, we have $z \in U$. Thus U is an ideal on S.

Now we can apply Zorn's Lemma and obtain a maximal element M in T. Finally, we show that M is a prime ideal. Every Boolean algebra is a distributive lattice. So by the theorem stating every maximal ideal is prime in a distributive lattice, M is a prime ideal.

□

As we previously mentioned, BPI is a weaker consequence of the Axiom of Choice. Moreover, BPI is equivalent to a weaker version of Tychonoff's Theorem.

Theorem 4.15. *The Boolean Prime Ideal Theorem implies Tychonoff's Theorem for Hausdorff Space.*

To prove this thereom, we need the Ultrafilter Lemma (stronger version), which is stated as follow:

Lemma 4.16. *Every filter on a set X is contained in an ultrafilter.*

Proof. Let F be an arbitrary filter on X. We will show that there exists an ultrafilter U such that $F \subseteq U$. Let A be the collection of all filters on X that contain F and partial order A by inclusion ($\subseteq$). Let C be an arbitrary chain in this set. Then $\cup C$ an upperbound for C because C is closed under finite intersections thus forms a base for a filter. It follows that every chain has an upperbound. By Zorn's Lemma, we have a maximal element of this set and name it U. This maximal element U is ultrafilter.

□

Lemma 4.17. *For any filter on S, the properties of being an ultrafilter or maximal are equivalent.*

Theorem 4.18. *The following are equivalent:*
(1) Boolean Prime Ideal Theorem
(2) The Ultrafilter Lemma
(3) Ultrafilter Convergence Theorem
(4) Tychonoff's Theorem for Hausdorff Spaces

(1) → (2)

Proof. Order the subsets of S by reverse inclusion (that is suppose $A, B \in S$, then $A \leq B$ if $A \supseteq B$). Then the result follows trivially from the Boolean Prime Ideal Theorem.

□

(2) → (3)

Proof. We prove the forward direction by contradiction. Let F be an ultrafilter on X such that X is compact but F does not converge to any point in X. Then by our assumption, for all $x \in X$ there must be some open neighborhood of x $U_x \notin F$, otherwise F converge to some x. Now we can rewrite X as the union of these open neighborhoods, that is $X = \cup_{x \in X} U_x$. However, since X is compact, we have $X = U_{x_1} \cup ... \cup U_{x_n}$. But $X \in F$ because F is an ultrafilter on X, which follows that there must be some $U_{x_i} \in F$, contradiction. Therefore F must have some limit points.

Conversely, suppose every ultrafilter on X converges to at least one point but X is not compact. Then there is an open cover $Y = \cup_i U_i$ on X with not finite subcover. So $\cap(Y \setminus U_i) = \emptyset$ but no finite intersection is empty or else it would give a finite subcover. Then $\{(Y \setminus U_i)\}$ has the finite intersection property thus it generates a filter, which then by the ultrafilter lemma can be extended to an ultrafilter F. Let x be the point that F converges to. But since every $(Y \setminus U_i) \in F$, by definition of ultrafilter, we know that $U_i \notin F$. It follows that the open neighborhood of x, $U_x \notin F$ as well. Then F does not converge to x, contradiction. Therefore X must be compact. □

(3) → (4)

Proof. Let $(X_i)_{i \in I}$ be a set of compact Hausdorff spaces and let $X = \Pi(X_i)_{i \in I}$ be the product of these Hausdorff spaces. Let U be an ultrafilter on X. Then for each $i \in I$, the set $U_i = \{A \subseteq X_i | \pi_i^{-1}[A] \in U\}$ is an ultrafilter on X_i.

Since X_i is compact and Hausdorff, every corresponding U_i converges to exactly one point x_i in X_i. We thus know that U converges to $x = (x_i)_{i \in I}$. By theorem 6, we know X is compact, as desired.

□

4.3 Stone-Cech Compactification

The property of beging compact is of great importance in topology. Many properties of a topological space are a lot easier to deduce if the space is a compact Hausdorff space. For example, compact Hausdorff spaces are normal (every two disjoint closed sets of X have disjoint open neighborhoods).

As a result, given a space X it is probably not compact and hence it is difficult to handle. The intuitive thing to do is therefore to look for a compact Hausdorff space K in which X can be embedded. We want K to be very similar to X; more specifically, we want a homeomorphism h going from X into a subspace h(X) of the compact Hausdorff space K.

Definition 4.19. Definition 1.1 Suppose $h : X \to Y$ is a homeomorphism of X into Y, where Y is a compact Hausdorff space. If $h(X)$ is dense in Y, then the pair (Y, h) is called a compactification of X.

In fact, the Stone-Cech compactification is the "largest" compactification of any Hausdorff space X. Let us see how the Stone-Cech Compactification is defined:

Definition 4.20. The Stone-Cech compactification of a set X is a compact Hausdorff space βX along with a map of sets $f : X \to \beta X$ satisfying the following universal property:

Given a compact Hausdorff space Y and a map of sets $\phi : X \to Y$, there is a unique continuous map $\omega : \beta X \to Y$ such that $\omega \circ f = \phi$

$$\begin{array}{ccc} X & \xrightarrow{f} & \beta X \\ & \searrow^{\phi} & \downarrow \omega \\ & & Y \end{array}$$

Construction of the Stone-Cech Compactification If X is discrete, one can construct βX as the set of all ultrafilters on X, with a topology known as the Stone topology. The elements of X correspond to the principal ultrafilters.

Underlying set: $\beta X = \{F : F \text{ is an ultrafilter on } X\}$

Topology: Given $A \subseteq X$, let $U_A = \{F : A \in F\}$. The sets U_A are a basis.

The universal map $f : x \to$ the principal ultrafilter generated by $\{x\}$.

Given $\phi : X \to Y$, define $\omega : \beta X \to Y$ by $F \to$ the F-limit of f.

5 Tychonoff's Theorem

Tychonoff's theorem is one of the most significant consequences of the Axiom of Choice in topology. Tychonoff's theorem is stated as follow:

Theorem 5.1. *$X = \prod X_{\alpha \in A}$ is compact if and only if (X_α, τ_α), for all $\alpha \in A$, are compact topological spaces.*

We will go over two proofs of this theorem. The first proof follows from Alexander's Subbase Theorem. We begin by proving Alexander's Subbase Theorem.

Lemma 5.2. *Alexander's Subbase Theorem*

Let (X, τ) be a topological space and ξ be a subbase for τ. If every covering of X by sets from ξ has a finite subcover, then X is compact.

Proof. We proceed by contradiction. Suppose that every cover from ξ has a finite subcover, but X is not compact. Let F be the collection of all open covers of X with no finite subcover partially ordered by inclusion.

We take an arbitrary nonempty chain $E \in F$. It follows that for any E_i and E_j in E, either $E_i \subseteq E_j$ or $E_j \subseteq E_i$. Let C be the union of all these covers. Obviously C is an upper bound of E. We next show that C is an element of F.

Since C is the union of all open sets in the chain E, it is clear that any finite collection of sets in C is an element in E because E is totally ordered by inclusion. Furthermore, every subcollection in E has no finite subcover. Hence the hypothesis of Zorn's Lemma is met.

By Zorn's Lemma, we have a maximal element M in F. In the next step we will show that the set $S = M \cap \xi$ does not cover X. In contrary, suppose that S covers X. Then by our assumption on ξ and the fact that $S \subseteq \xi$, we know S has a finite subcover for X. However, it follows that this finite subcover is also in M, which contradicts our assumption that M is an element of F and does not have a finite subcover for X. Therefore $M \cap \xi$ does not cover X.

Thus there is some $x \in X$ with $x \notin S$. But x is covered by M so that $x \in U$ for some $U \in M$.

Also, since ξ is a subbasis, we know that $x \in B_1 \cap ... \cap B_n \subseteq U$ for some $B_1, ..., B_n \in \xi$. Moreover, we have $B_i \notin M$ for all $i = 1, ..., n$ because otherwise some $B_i \in M \cap \xi$ and S covers x, which contradicts our assumption. Then due to the maximality of M, (so M is the largest set that has no finite subcover; any set bigger than M should have a finite subcover) we can find a corresponding finite subset M_{B_i} of M such that $M_{B_i} \cup B_i$ forms a finite cover of X. Let $M_F = M_{B_1} \cup ... \cup M_{B_n}$. Then for each i, the former finite cover of X can be replaced by this new larger finite cover $M_F \cup B_i$. Further, since the finite set $M_F \cup B_i$ covers X, $(B_1 \cap ... \cap B_n) \cup M_F$ is also a finite cover of X. However, then $B_1 \cap ... \cap B_n$ can be replaced by a single open set in M, as shown previously. Therefore $U \cup M_F$ is a finite cover of X, which solely contains elements from M. This result contradicts our assumption on M.

Finally we can conclude that X is compact. □

Now we prove Tychonoff's theorem.

Proof. Let X=$\Pi_{a\in A}X_\alpha$, where each X_α is compact. Let D be a family of subbasic sets of the form $\pi_\alpha^{-1}(U_\alpha)$, where $\alpha \in A$ and U_α is an open subset of X_α.

Suppose that no finite subfamily of D covers X. In view of Alexander's subbase theorem, it suffices to show that D does not cover X.

Fix an index β and consider the subbasic sets in D of the form $\pi_\alpha^{-1}(V)$, where V is an open subset of X_β. The aggregate of such open sets V cannot cover X_β. Choose a point $x(\beta) \in X_\beta$ such that $x(\beta)$ is not included in the union of the Vs. The various choices $x(\beta)$ determine a point $x \in \Pi X_\alpha$, which is evidently not included in any of the sets in D, as desired. □

The second proof uses ultrafilters. We begin this part by introducing some related definitions and lemmas.

Definition 5.3. Let F be a filter on a set X and let $f : X \to Y$. We define the pushforward $f * F$ to be the family $\{A \subseteq Y : f^{-1}(A) \in F\}$.

Lemma 5.4. *Let F be an ultrafilter on the set X and let $f : X \to Y$ be a map of sets. Then $f_*F = \{A \subseteq Y : f^{-1}(A) \in F\}$ is an ultrafilter on Y.*

Now we look at the proof of Tychonoff's Theorem.

Proof. Let $X_i, i \in I$ be a collection of compact topological spaces and let $X = \Pi_{i\in I}X_i$ with projection maps $\pi_i : X \to X_i$. We will show that every ultrafilter on X converges to at least one point.

By lemma 5.6, we know that $(\pi_i)_*F$ is a collection of ultrafilters on X_i. By our assumption that every set X_i is compact, we know that every ultrafilter on these sets converge to some point $x_i \in X_i$. We claim that F converge to $x =< x_i >_{i\in I} \in X$.

Now we consider the topology on X generated by sets V of the form $V = \pi^{-1}(U_i)$, where $U_i \subseteq X_i$ is an open set containing x_i. So any open set containing x contains a finite intersection of sets of this form which contain x. Since F is closed under supersets and finite intersections (by definition of filter), it suffices to show that if $x \in V = \pi^{-1}(U_i)$, then $V \in F$.

Suppose $x \in V$. Then $x_i \in U$, so $U \in (\pi_i)_*F$, since $(\pi_i)_*F$ converges to x_i. But then by definition of $(\pi_i)_*F$, $V \in F$. This completes the proof. □

Theorem 5.5. *Tychonoff's Theorem implies the Axiom of Choice.*

Kelley's Proof in 1950s

Proof. We will show that if for each $a \in A$, X_a is not empty, then the product $\Pi_{a \in A} X_a$ is not empty.

We begin by adding a single point s to each of the set X_a. Let $Y_a = X_a \cup \{s\}$. We assign a topology for Y_a by defining the empty set and complements of finite sets to be open (equip Y_a with the cofinite topology), and show that this cofinite topology on Y_a is compact.

Suppose U is an open cover in this cofinite topology. Take an arbitrary nonempty set from this cover. This set is cofinite, therefore only finitely many point are not in this open set. This open set together with sets containing these finitely many points form a finite cover for this topology on Y_a. Then by Tychonoff Theorem, we know that $\Pi_{a \in A} Y_a$ is compact.

For each $a \in A$, let Z_a be that subset of $\Pi_{a \in A}\ Y_a$ consisting of all points whose coordinate lies in X_a ($Z_a = X_a * \Pi_{d \in A} Y_d$ with $d \neq a$). Surely Z_a is closed in $\Pi_{a \in A} Y_a$ since X_a is closed in Y_a. (why: if X_a is closed in Y_a then Z_a is closed in $\Pi_{a \in A} Y_a$.)

Moreover, for any finite subset B of A, the intersection $\cap_{a \in B} Z_a$ is not empty, because each X_a is non-empty. By the finite axiom of choice, we can choose $x_a \in X_a$ for $a \in B$, and set $x_a = s$ for $a \in A - B$. Consequently, the family of all sets of the form Z_a (for some $a \in A$), is a family of closed subsets of $\Pi_{a \in A} Y_a$, with the property that the intersection of any finite subfamily is not empty. And since $\Pi_{a \in A} Y_a$ is compact, the intersection $\cap_{a \in A} Z_a$ is not empty, and the axiom of choice is proved.

□

One Minor Mistake in Kelley's Proof However, Kelley's proof is flawed. According to his assumptions, X_a is open rather than closed in Y_a. Without further assumptions, the set of the single point s is finite, thus the complement of s, X_a is open in this cofinite topology.

Correction

Proof. Let $X = \Pi X_{a \in A}$ be a product space. Add one point s to every X_a and let $Y_a = X_a \cup \{s\}$. To correct this proof, we modify the topology τ_a assigned to Y_a for $a \in A$. We assign τ_a to be the cofinite topology and further declare that $\{s\}$ is open in the topology.

Again any cofinite topology is compact so X_a is compact (we have shown this in Kelley's proof). Additionally, adding a single point to a compact space preserve compactness. All we do is add in one open set to the finite cover of X_a that contains the single point s. Therefore Y_a is compact. By Tychonoff's Theorem, we know that $Y = \Pi_{a\in A} Y_a$ is compact as well. Next we will show that $\Pi_{a\in A} X_a$ is not empty.

Because we assume that s is open in Y_a, X_a is closed in Y_a. Moreover, since the projection $\pi_a : X \twoheadrightarrow X_a \cup \{s\}$ is continuous, we know that $\pi_a^{-1}(X_a)$ is closed in X. We define a new set $F = \{\pi_a^{-1}(X_a) | a \in A\}$, containing all preimage of X_a with $a \in A$.

First, we use induction to show that F has the finite intersection property. Pick arbitrary sets $F_1, ..., F_n \in F$ such that $F_i = \pi_i^{-1}(X_i)$ for all $i = 1, ..., n$. We will show that there exists $d \in F_1 \cap ... \cap F_n$. Since each X_a is not empty, we can always pick $x_a \in X_a$ for all X_a. We define d as follow: $d_i = x_{a_i}$ if $i = a_i \; for \; some \; x_{a_i} \in X_a$; otherwise let $d_i = s$. Then $d \in F$ and FIP holds for F.

By the compactness of X, there is some $x \in X$ that belongs to every member of F. In other word, there exists $x \in \cap_{a\in A} \pi_a^{-1}(X_a)$. We have shown that $\Pi_{a\in A} X_a$ is not empty, as desired. □

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