Applications of Covering Sets

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Objectives

1. History behind Covering Sets
2. Mathematical Background
3. Example Coverings
4. What I did for Research
Motivations

Polignac Conjecture - 1849

Every odd integer $k$ can be written as $2^n + p$, where $p$ is an odd prime and $n$ is a positive integer.
Motivations

**Polignac Conjecture - 1849**
Every odd integer $k$ can be written as $2^n + p$, where $p$ is an odd prime and $n$ is a positive integer.

**Counterexamples**
Polignac’s Conjecture was easily proved to be false via 127 and 509, but the problem was not completely discarded and provoked further thought into the subject.
Paul Erdős, 20th century Hungarian mathematician.

- Introduced and developed the theory behind covering sets.
- Proved there are infinitely many counterexamples to Polignac’s conjecture.
- Covering sets are now used to explore variations of the Polignac conjecture, some involving the Fibonacci numbers due to the special properties they possess.
Definition

A finite covering set of the integers is a system of congruences \( n \equiv r_i \mod m_i \), with \( 1 \leq i \leq t \), such that every integer \( n \) satisfies at least one of the congruences. To avoid trivial solutions we want \( m_i > 1 \) for all \( i \).
Covering Sets

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Example

A complete residue system mod $p$.

\[
\begin{align*}
    n &\equiv 0 \mod p \\
    n &\equiv 1 \mod p \\
    &\vdots \\
    n &\equiv p - 1 \mod p
\end{align*}
\]

We can write this covering set as $\{(0, p), (1, p), \ldots, (p - 1, p)\}$. 
Extra Conditions

We are building these sets as a system of triples \((r_i, m_i, p_i)_{i=1}^{t}\) with the properties

1. The set \(\{(r_i, m_i)\}_{i=1}^{t}\) is a covering set of the integers.
2. \(p_1, p_2, ..., p_t\) are all distinct odd primes which play an auxiliary role.
3. \(L = \prod_{i=1}^{t} m_i\) is the least common multiple (LCM) of the covering.
Examples of Covering Sets

Polignac Problem

Do there exist infinitely many positive integers $k$ such that $k - 2^n$ is composite for all integers $n \geq 0$?

Erdős's Covering for $2^n \pm p$

Erdős used a covering set in which $\text{ord}_{p^i}(2) = m_i$, where $\text{ord}_{p^i}(2)$ denotes the multiplicative order of 2 modulo $p$, the smallest positive integer $m_i$ such that $2^{m_i} \equiv 1 \pmod{p}$.
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\[
\{ (0, 2, 3), (0, 3, 7), (1, 4, 5), (3, 8, 17), (7, 12, 13), (23, 24, 241) \}
\]
How Does the Covering Work

Polignac Problem

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What Each Triple Means

- Take the triple $(1, 4, 5)$ in the covering.
- $4 = \text{ord}_5(2)$ so $2^4 \equiv 1 \mod 5$.
- When $n \equiv 1 \mod 4$ we want $k - 2^n \equiv 0 \mod 5$.
- $k - 2^1 \equiv 0 \mod 5 \implies k \equiv 2 \mod 5$. 

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How Does the Covering Work

Full System of Congruences

\[ \begin{align*}
    k &\equiv 1 \mod 3, & k &\equiv 1 \mod 7, & k &\equiv 2 \mod 5 \\
    k &\equiv 8 \mod 17, & k &\equiv 11 \mod 13, & k &\equiv 121 \mod 241
\end{align*} \]

Finding a Solution

Now that we have a system of congruences for \( k \) we apply the Chinese Remainder Theorem to the system, with \( k \equiv 1 \mod 2 \) as well, to get infinitely many solutions \( k \equiv 7629217 \mod 11184810 \).
Fibonacci Numbers

The Fibonacci numbers are a sequence of numbers where \( F_0 = 0 \), \( F_1 = 1 \) and for all \( n \geq 2 \), \( F_n = F_{n-1} + F_{n-2} \).

\[
F = \{0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, \ldots\}
\]
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$$F = \{0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, \ldots\}$$

Pisano Period

The $n^{th}$ Pisano Period, denoted by $\pi(n)$, is the period with which the sequence of Fibonacci Numbers modulo $n$ repeats.

Example

Let $F$ be the Fibonacci Numbers, so $F = \{0, 1, 1, 2, 3, \ldots\}$. Then $\pi(2) = 3$ since $F \mod 2 = \{0, 1, 1, 0, 1, 1, \ldots\}$. 
Jones’s Covering

In 2012, Lenny Jones proved there are infinitely many positive integers that cannot be written as $F_n \pm p$ where $F_n$ is the $n^{th}$ Fibonacci number and $p$ is a prime.
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$$C = \{(0, 3, 2), (0, 8, 3), (1, 10, 11), (6, 14, 29), (6, 16, 7), (5, 18, 19), (3, 20, 5), (2, 28, 13), (19, 30, 31), (12, 32, 47), (29, 36, 17), (27, 40, 41), (22, 42, 211), (20, 48, 23), (5, 50, 101), (45, 50, 151), (35, 54, 5779), (18, 56, 281), (37, 60, 61), (0, 70, 71), (12, 70, 911), (47, 72, 107), (14, 80, 2161), (10, 84, 421), (89, 90, 181), (85, 90, 541), (92, 96, 1103), (13, 100, 3001), (53, 108, 53), (17, 108, 109), (42, 112, 14503), (7, 120, 2521), (40, 126, 1009), (124, 126, 31249), (42, 140, 141961), (100, 144, 103681), (85, 150, 12301), (115, 150, 18451), (78, 160, 1601), (46, 160, 3041), (50, 162, 3079), (140, 162, 62650261), (122, 168, 83), (50, 168, 1427), (73, 180, 109441), (75, 200, 401), \}
(175, 200, 570601), (110, 210, 21211), (196, 210, 767131),
(4, 216, 11128427), (158, 224, 10745088481), (193, 240, 241),
(133, 240, 20641), (82, 252, 35239681), (29, 270, 271), (17, 270, 811),
(119, 270, 42391), (209, 270, 119611), (154, 280, 12317523121),
(28, 288, 10749957121), (25, 300, 230686501), (124, 324, 2269),
(232, 324, 4373), (148, 324, 19441), (26, 336, 167), (206, 336, 65740583),
(98, 350, 54601), (168, 350, 560701), (28, 350, 7517651),
(238, 350, 51636551), (133, 360, 10783342081), (88, 378, 379),
(130, 378, 85429), (214, 378, 912871), (52, 378, 1258740001),
(393, 400, 9125201), (153, 400, 5738108801), (278, 420, 8288823481),
(292, 432, 6263), (196, 432, 177962167367), (215, 450, 221401),
(35, 450, 15608701), (335, 450, 3467131047901),
(446, 480, 23735900452321), (268, 504, 1461601), (436, 504, 764940961),
(107, 540, 1114769954367361), (306, 560, 118021448662479038881),
(73, 600, 601), (433, 600, 87129547172401), (92, 630, 631),
(476, 630, 1051224514831), (260, 630, 1983000765501001),
(340, 648, 1828620361), (364, 648, 6782976947987),
(638, 672, 115613939510481515041), (658, 700, 701),
(474, 700, 17231203730201189308301), (13, 720, 8641),
(515, 720, 13373763765986881), (700, 756, 38933),
(472, 756, 955921950316735037), (715, 800, 124001), (315, 800, 6996001),
(782, 800, 3160438834174817356001), (742, 810, 1621), (94, 810, 4861),
(580, 810, 21871), (418, 810, 33211), (256, 810, 31603395781),
(34, 810, 7654861102843433881), (194, 840, 721561),
(266, 840, 140207234004601), (508, 864, 3023), (412, 864, 19009),
(14, 864, 447901921), (686, 864, 48265838239823),
(242, 900, 11981661982050957053616001), (46, 1008, 503),
(494, 1008, 4322424761927), (830, 1008, 571385160581761),
(302, 1050, 1051), (722, 1050, 934645940780547345401),
(512, 1050, 14734291702642871390242051), (590, 1080, 12315241),
(950, 1080, 100873547420073756574681), (942, 1120, 6135922241),
(270, 1120, 164154312001), (750, 1120, 13264519466034652481),
(428, 1134, 89511254659), (680, 1134, 1643223059479),
(806, 1134, 68853479653802041437170359),
(1058, 1134, 5087394106095783259).
BJLM Theorem

There exist infinitely many positive integers $k$ such that

$$k(2^n + F_n) \pm 1$$

is composite for all integers $n \geq 0$.

$$C = \{(1,3,2), (2,3,2), (0,4,3), (3,8,3), (5,8,3), (2,10,11), (9,18,19), (15,18,19), (9,20,5), (14,20,5), (3,30,31), (6,30,31), (18,40,41), (33,48,7), (6,48,7), (21,60,61), (30,60,61), (39,72,17), (165,180,181), (57,240,241), (78,240,241)\}$$

contains 21 ordered triples, with $L = 720$. 
**Problem**

Do there exist infinitely many positive integers $k$ such that $k^2(2^n + F_n) + 1$ is composite for all integers $\geq 0$?

**Strategy**

Finding a covering with the following requirements:

1. $p_i$ divides $2^{m_i} - 1$,
2. $m_i$ is a multiple of $\pi(p_i)$,
3. $(2^n + F_n) \not\equiv 0 \mod p$,
4. $-(2^n + F_n)^{-1}$ to be a Quadratic Residue mod $p$. 
New Problem

\[ C = \{(1,3,2),(2,3,2),(18,24,3),(2,8,3),(0,20,5),(7,20,5),(9,20,5), (14,20,5),(1,48,7),(3,48,7),(14,48,7),(22,48,7),(23,48,7),(26,48,7), (31,48,7),(34,48,7),(41,48,7),(45,48,7),(6,10,11),(6,72,17),(18,72,17), (23,72,17),(33,72,17),(40,72,17),(43,72,17), (9,18,19),(15,18,19), (5,30,31), (25,30,31),(27,30,31),(2,40,41), (11,40,41),(33,40,41), (21,60,61),(30,60,61),(60,180,181),(99,180,181),(138,180,181), (163,180,181),(12,240,241),(83,240,241), (105,240,241),(108,240,241),(143,240,241)\} \]
Approaching the Problem

What I Tried

- Choose $L$ early in the problem solving process.
- Smart choice of $r_i$, avoid 1 where possible.

\[ F \text{ mod } n = \{0, 1, 1, \ldots, 1\} \]

- Use computer programs to help determine if a given list of triples is a covering, what $r_i$ values occur most frequently.

Obstacles

- No closed formula for Pisano Periods.
- Do not want a covering to be too large.

Looking Forward

- Create a program to search for out the $r_i, m_i$ for a given $L$ and $p$. 
Thank You