Applications of Covering Sets

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Objectives

1. History behind Covering Sets
2. Mathematical Background
3. Example Coverings
4. What I did for Research

Motivations

Polignac Conjecture - 1849

Every odd integer k can be written as $2^n + p$, where p is an odd prime and n is a positive integer.

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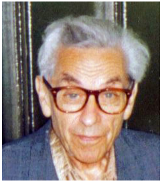
Counterexamples

Polignac's Conjecture was easily proved to be false via 127 and 509, but the problem was not completely discarded and provoked further thought into the subject.

Erdős



- Paul Erdős, 20^{th} century Hungarian mathematician.
- Introduced and developed the theory behind covering sets.
- Proved there are infinitely many counterexamples to Polignac's conjecture.
- Covering sets are now used to explore variations of the Polignac conjecture, some involving the Fibonacci numbers due to the special properties they possess.

Covering Sets

Definition

A finite covering set of the integers is a system of congruences $n \equiv r_i \mod m_i$, with $1 \leq i \leq t$, such that every integer n satisfies at least one of the congruences. To avoid trivial solutions we want $m_i > 1$ for all i.

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Example

A complete residue system mod p.

$$\begin{aligned} n &\equiv 0 \mod p \\ n &\equiv 1 \mod p \\ &\vdots \\ n &\equiv p-1 \mod p \end{aligned}$$

We can write this covering set as $\{(0,p),(1,p),\ldots,(p-1,p)\}$.

Extra Conditions

We are building these sets as a system of triples $(r_i, m_i, p_i)_{i=1}^{i=t}$ with the properties

1. The set $\{(r_i, m_i)\}_{i=1}^{i=t}$ is a covering set of the integers.
2. $p_1, p_2, ..., p_t$ are all distinct odd primes which play an auxiliary role.
3. $L = \prod_{i=1}^{i=t} m_i$ is the least common multiple (LCM) of the covering.

Examples of Covering Sets

Polignac Problem

Do there exist infinitely many positive integers k such that $k - 2^n$ is composite for all integers $n \geq 0$?

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Erdős's Covering for $2^n \pm p$

Erdős used a covering set in which $ord_{p_i}(2) = m_i$, where $ord_p(2)$ denotes the multiplicative order of 2 modulo p, the smallest positive integer m such that $2^m \equiv 1 \mod p$.

$$\{(0,2,3),(0,3,7),(1,4,5),(3,8,17),(7,12,13),(23,24,241)\}$$

How Does the Covering Work

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What Each Triple Means

- Take the triple $(1,4,5)$ in the covering.
- $4 = ord_5(2)$ so $2^4 \equiv 1 \bmod 5$.
- When $n \equiv 1 \bmod 4$ we want $k - 2^n \equiv 0 \bmod 5$.
- $k - 2^1 \equiv 0 \bmod 5 \implies k \equiv 2 \bmod 5$.

How Does the Covering Work

Full System of Congruences

$$k \equiv 1 \bmod 3, \quad k \equiv 1 \bmod 7, \quad k \equiv 2 \bmod 5$$
$$k \equiv 8 \bmod 17, \quad k \equiv 11 \bmod 13, \quad k \equiv 121 \bmod 241$$

Finding a Solution

Now that we have a system of congruences for k we apply the Chinese Remainder Theorem to the system, with $k \equiv 1 \mod 2$ as well, to get infinitely many solutions $k \equiv 7629217 \mod 11184810$.

Fibonacci

Fibonacci Numbers

The Fibonacci numbers are a sequence of numbers where $F_0 = 0, F_1 = 1$ and for all $n \geq 2, F_n = F_{n-1} + F_{n-2}$.

$$F = \{0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, ...\}$$

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$$F = \{0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, ...\}$$

Pisano Period

The n^{th} Pisano Period, denoted by $\pi(n)$, is the period with which the sequence of Fibonacci Numbers modulo n repeats.

Example

Let F be the Fibonacci Numbers, so $F = \{0, 1, 1, 2, 3, ...\}$. Then $\pi(2) = 3$ since $F \mod 2 = \{0, 1, 1, 0, 1, 1, ...\}$.

Examples of Covering Sets

Jones's Covering

In 2012, Lenny Jones proved there are infinitely many positive integers that cannot be written as $F_n \pm p$ where F_n is the n^{th} Fibonacci number and p is a prime.

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$$
\begin{aligned}
C = \{&(0,3,2),(0,8,3),(1,10,11),(6,14,29),(6,16,7),(5,18,19),\\
&(3,20,5),(2,28,13),(19,30,31),(12,32,47),(29,36,17),\\
&(27,40,41),(22,42,211),(20,48,23),(5,50,101),(45,50,151),\\
&(35,54,5779),(18,56,281),(37,60,61),(0,70,71),(12,70,911),\\
&(47,72,107),(14,80,2161),(10,84,421),(89,90,181),(85,90,541),\\
&(92,96,1103),(13,100,3001),(53,108,53),(17,108,109),\\
&(42,112,14503),(7,120,2521),(40,126,1009),(124,126,31249),\\
&(42,140,141961),(100,144,103681),(85,150,12301),(115,150,18451),\\
&(78,160,1601),(46,160,3041),(50,162,3079),(140,162,62650261),\\
&(122,168,83),(50,168,1427),(73,180,109441),(75,200,401),
\end{aligned}
$$

Jones continued

$(175, 200, 570601), (110, 210, 21211), (196, 210, 767131),$
$(4, 216, 11128427), (158, 224, 10745088481), (193, 240, 241),$
$(133, 240, 20641), (82, 252, 35239681), (29, 270, 271), (17, 270, 811),$
$(119, 270, 42391), (209, 270, 119611), (154, 280, 12317523121),$
$(28, 288, 10749957121), (25, 300, 230686501), (124, 324, 2269),$
$(232, 324, 4373), (148, 324, 19441), (26, 336, 167), (206, 336, 65740583),$
$(98, 350, 54601), (168, 350, 560701), (28, 350, 7517651),$
$(238, 350, 51636551), (133, 360, 10783342081), (88, 378, 379),$
$(130, 378, 85429), (214, 378, 912871), (52, 378, 1258740001),$
$(393, 400, 9125201), (153, 400, 5738108801), (278, 420, 8288823481),$
$(292, 432, 6263), (196, 432, 177962167367), (215, 450, 221401),$
$(35, 450, 15608701), (335, 450, 3467131047901),$
$(446, 480, 23735900452321), (268, 504, 1461601), (436, 504, 764940961),$
$(107, 540, 1114769954367361), (306, 560, 118021448662479038881),$
$(73, 600, 601), (433, 600, 87129547172401), (92, 630, 631),$
$(476, 630, 1051224514831), (260, 630, 1983000765501001),$

Jones continued

$(340, 648, 1828620361), (364, 648, 6782976947987),$
$(638, 672, 1156139395104815150 41), (658, 700, 701),$
$(474, 700, 172312037302011893083 01), (13, 720, 8641),$
$(515, 720, 1337376376598688 1), (700, 756, 38933),$
$(472, 756, 9559219503167350 37), (715, 800, 124001), (315, 800, 6996001),$
$(782, 800, 3160438834174817356001), (742, 810, 1621), (94, 810, 4861),$
$(580, 810, 21871), (418, 810, 33211), (256, 810, 31603395781),$
$(34, 810, 7654861102843433881), (194, 840, 721561),$
$(266, 840, 140207234004601), (508, 864, 3023), (412, 864, 19009),$
$(14, 864, 447901921), (686, 864, 482658382398 23),$
$(242, 900, 1198166198205095705361 6001), (46, 1008, 503),$
$(494, 1008, 4322424761927), (830, 1008, 571385160581761),$
$(302, 1050, 1051), (722, 1050, 9346455940780547345401),$
$(512, 1050, 1473429170264287139024205 1), (590, 1080, 12315241),$
$(950, 1080, 1008735474200737565746 81), (942, 1120, 6135922241),$
$(270, 1120, 164154312001), (750, 1120, 1326451946603465248 1),$
$(428, 1134, 89511254659), (680, 1134, 1643223059479),$
$(806, 1134, 6885347965380204143717035 9),$
$(1058, 1134, 508739410609578325 9)\}.$

BJLM Theorem

There exist infinitely many positive integers k such that

$$k(2^n + F_n) \pm 1$$

is composite for all integers $n \geq 0$.

$$\begin{aligned} C = \quad & \{(1,3,2),(2,3,2),(0,4,3),(3,8,3),(5,8,3),(2,10,11),(9,18,19), \\ & (15,18,19),(9,20,5),(14,20,5),(3,30,31),(6,30,31),(18,40,41),(33,48,7), \\ & (6,48,7),(21,60,61),(30,60,61),(39,72,17),(165,180,181), \\ & (57,240,241),(78,240,241)\} \end{aligned}$$

contains 21 ordered triples, with $L = 720$.

New Problem

Problem

Do there exist infinitely many positive integers k such that $k^2(2^n + F_n) + 1$ is composite for all integers ≥ 0?

Strategy

Finding a covering with the following requirements:

1. p_i divides $2^{m_i} - 1$,
2. m_i is a multiple of $\pi(p_i)$,
3. $(2^n + F_n) \not\equiv 0 \mod p$,
4. $-(2^n + F_n)^{-1}$ to be a Quadratic Residue mod p.

New Problem

C = {(1,3,2),(2,3,2),(18,24,3),(2,8,3),(0,20,5),(7,20,5),(9,20,5),
(14,20,5),(1,48,7),(3,48,7),(14,48,7),(22,48,7),(23,48,7),(26,48,7),
(31,48,7),(34,48,7),(41,48,7),(45,48,7),(6,10,11),(6,72,17),(18,72,17),
(23,72,17),(33,72,17),(40,72,17),(43,72,17), (9,18,19),(15,18,19),
(5,30,31), (25,30,31),(27,30,31),(2,40,41), (11,40,41),(33,40,41),
(21,60,61),(30,60,61),(60,180,181),(99,180,181),(138,180,181),
(163,180,181),(12,240,241),(83,240,241),
(105,240,241),(108,240,241),(143,240,241)}

Approaching the Problem

What I Tried

- Choose L early in the problem solving process.
- Smart choice of r_i, avoid 1 where possible.

$$F \bmod n = \{0, 1, 1, ..., 1\}$$

- Use computer programs to help determine if a given list of triples is a covering, what r_i values occur most frequently.

Obstacles

- No closed formula for Pisano Periods.
- Do not want a covering to be too large.

Looking Forward

- Create a program to search for out the r_i, m_i for a given L and p.

Thank You